

Thermodynamic Efficiency of Feynman-Smoluchowski Ratchet: Unified View through Explicit Coarse-Graining

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Coarse-graining the description of a fluctuating heat engine can change its apparent dissipative feature and affect the estimated efficiency. We here explicitly show that the Feynman-Smoluchowski (FS) ratchet, a classical thought experiment describing an autonomous heat-work converter, suffers from this very problem of hidden dissipation. Through singular perturbation analysis of the original FS ratchet model, we obtain the exact form of the coarse-grained dynamics, which turns out to be the Büttiker-Landauer motor, another classical model of Brownian heat engine. Explicit forms of the hidden entropy productions provide a unified understanding on how the original argument by Feynman led to the overestimated efficiency. The approach serves as a key example of how we can systematically address the problem of thermodynamic efficiency in a multi-variable fluctuating system.

Introduction.— The framework of stochastic thermodynamics has not only allowed experimental characterization of small thermodynamic systems[1–3], but has also established a unified scheme to address fundamental questions in thermodynamics. Identities and inequalities formulated for general stochastic dynamics have been given thermodynamical interpretations such as the second law[4–6], role of information feedback[7, 8], bound on efficiencies of engines at finite time operations[9, 10], and laws extended to nonequilibrium setups[11–13].

The crucial concept behind the developments in stochastic thermodynamics is the entropy production, which is typically defined through local detailed balance using transition probabilities [14]. It has been pointed out, however, that this entropy production is only an effective quantity, meaning that it does not always correspond to the physical entropy production that appears in conventional thermodynamics [15–17]. This is clearly illustrated in the situation where the system may be modeled at different scales of description; the entropy production calculated from the fine-grained transition probabilities can be different from that obtained using coarse-grained transition probabilities. In this case, even if the original system correctly described the thermodynamic nature of the system, the coarse-graining step can “hide” a part of the entropy production, leaving only an entropy production-like quantity in the coarse-grained system.

In the classical example of the Feynman-Smoluchowski (FS) ratchet, we notice that a similar issue to this scale-dependent entropy production has been discussed. The FS ratchet is a thought experiment where the ratchet and pawl attached to two different heat baths mechanically interact (see FIG. 1a). Smoluchowski argued that there is no rotation and extracted work if the device is placed in an isothermal environment [18]. Feynman revisited this problem in his famous textbook [19] to consider the case where the temperatures are different in the two baths, and claimed that the ratchet can operate as a Carnot-efficient engine at the stalled state, where the driving force balances with the load. Feynman’s argument

was based on the evaluation of exchanged energy with the heat baths at forward/backward steps of rotation, which is essentially simplifying the original dynamics to only focus on the discrete relative position of the ratchet. Parrondo and Español found, however, that there is a finite heat flow through the kinetic energy even at the stalled state [20], meaning that the momentum variable overlooked by Feynman’s analysis plays a significant role in the dissipation. Numerous studies based on specific simplified models [21–23] followed up on the existence of such heat leak, essentially proving that the FS ratchet cannot operate at Carnot efficiency.

In this Letter, we show explicitly that the classical discussions in the efficiency of the FS ratchet can be restated as a problem of hidden entropy production, i.e., the discrepancy of irreversible entropy production in the different scales of descriptions. We derive two coarse-grained description of the FS ratchet (Model-2 and Model-3), and exact relations between the dissipative features of different descriptions [Eqs. (7,8)] through singular perturbation analysis. Numerical simulations clarify the impact of hidden entropy production on the thermodynamic efficiencies (FIG. 4). Our results provide a unified view on the efficiency of the FS ratchet model, and illustrate the utility of the controlled perturbation method in addressing thermodynamic behaviors of multi-variable fluctuating engines.

Setup and Coarse-graining.— As shown in FIG. 1a, the FS ratchet consists of a vane and a gear connected by a rigid axle, and a pawl meshing with the gear. An external load couples with the axle, and applies a constant torque, f . The equations of motion for the angle of the coaxial vane and gear θ and the height of the pawl x reads

$$\begin{aligned}\dot{\theta} &= \frac{p}{m}, \\ \dot{p} &= -\frac{\Gamma}{m}p + f - \frac{\partial U(\theta, x)}{\partial \theta} + \sqrt{2\Gamma T_h}\xi, \\ \gamma \dot{x} &= -\frac{\partial U(\theta, x)}{\partial x} + \sqrt{2\gamma T_c}\zeta,\end{aligned}\quad (\text{Model-1})$$

with $U(\theta, x) = \lambda[x - \phi(\theta)]^2/2 + kx^2/2$ for simplicity (see [25]). Here, p is the angular momentum conjugated to θ , and m is the corresponding moment of inertia. The effect of two heat baths are modeled as Langevin heat baths characterized by the temperatures, T_h and T_c , and the viscous frictional coefficients, Γ and γ . ξ and ζ are independent white Gaussian noises with zero means and unit variances. We have already eliminated the momentum variable of the pawl; this overdamped limit coarse-graining is trivial and does not involve hidden entropy production [25]. The first and second terms of mechanical potential $U(\theta, x)$ are attributed to the spring and the interaction between the tip of pawl and the surface of gear, respectively. The periodic function $\phi(\theta)$ describes the shape of gear. k and λ are the spring coefficient and the strength of interaction, respectively.

We first consider the limit where the tip of pawl is tightly confined on the surface of gear: $\lambda/\gamma \rightarrow \infty$. From singular perturbation analysis [25], we obtain the coarse-grained equation of motion

$$\begin{aligned} \dot{\theta} &= \frac{p}{m}, \\ \dot{p} &= -\frac{\mathcal{G}(\theta)}{m}p + f - \frac{\partial U_{\text{eff}}(\theta)}{\partial \theta} + \sqrt{2\mathcal{G}(\theta)T_{\text{eff}}(\theta)}\Xi, \end{aligned} \quad (\text{Model-2})$$

where Ξ represents another white Gaussian noise with zero mean and unit variance. As shown in FIG. 1b, this dynamics can be identified as the underdamped Langevin equation with a single degree of freedom under the effective potential $U_{\text{eff}}(\theta) = k\phi(\theta)^2/2$. The key finding is that the derived effective frictional coefficient \mathcal{G} and the effective temperature T_{eff}

are functions of θ :

$$\mathcal{G}(\theta) = \Gamma + \gamma\phi'(\theta)^2, \quad (1)$$

$$T_{\text{eff}}(\theta) = \frac{\Gamma T_h + \gamma\phi'(\theta)^2 T_c}{\Gamma + \gamma\phi'(\theta)^2}. \quad (2)$$

From Model-2, we can further take the overdamped limit $m/\Gamma \rightarrow 0$, while keeping the ratio γ/Γ constant, again through singular perturbation theory. We obtain overdamped Langevin equation [1, 25]

$$\mathcal{G}(\theta) \odot \dot{\theta} = f - \frac{\partial U_{\text{eff}}(\theta)}{\partial \theta} + \sqrt{2\mathcal{G}(\theta)} \odot \sqrt{T_{\text{eff}}(\theta)} \cdot \Xi. \quad (\text{Model-3})$$

The symbols \cdot and \odot indicate the product in the sense of Itô and anti-Itô, respectively, which specify the interpretation of factor on the left:

$$\sqrt{2\mathcal{G}(\theta)} \odot \sqrt{T_{\text{eff}}(\theta)} \cdot \Xi = \lim_{\Delta t \rightarrow 0} \sqrt{2\mathcal{G}(\theta_{t+\Delta t})T_{\text{eff}}(\theta_t)} \frac{1}{\Delta t} \int_t^{t+\Delta t} \Xi_s ds. \quad (3)$$

We show in [25] that we arrive at a different overdamped model when switching the order of the tightly confined and overdamped limits.

Through these coarse-grained dynamics, we can observe how the FS ratchet operates as an autonomous heat engine. Firstly, it is clear from Eq. (2) that in the case of $T_h = T_c$ the effective temperature will become constant $T_{\text{eff}} = T_h = T_c$ leading to isothermal dynamics producing no positive work $W = -f\langle\dot{\theta}\rangle \leq 0$. Here, W is the steady-state work rate, and $\langle\cdot\rangle$ represents the steady-state ensemble average. According to Model-3, in the case where $T_h \neq T_c$, the θ -dependent effective temperature can produce work, $W > 0$, if the generalized potential difference $-\oint d\theta U'_{\text{eff}}(\theta)/T_{\text{eff}}(\theta)$ has sufficiently large magnitude and an opposite sign to f . This phenomena is known as the Büttiker-Landauer (BL) motor [24]. Coarse-graining and the thermodynamic efficiency of the BL motor (i.e., from underdamped to overdamped) has been discussed previously [15, 26, 27], and the details of the efficiency will be discussed in the next section. The coarse-grained descriptions #2 and #3 are the first main results of this Letter.

Stochastic Thermodynamics of the FS ratchet.— The standard prescription of stochastic thermodynamics [2] gives the effective entropy production rates in the heat baths through the transition probabilities of the models. We introduce the heat fluxes $Q_1^h = -(\dot{p} + \partial U/\partial \theta - f) \odot p/m$, $Q_1^c = -\partial U/\partial x \odot \dot{x}$, $Q_2 = -(\dot{p} + \partial U_{\text{eff}}/\partial \theta - f) \odot p/m$ and $Q_3 = [-\partial(U_{\text{eff}} + T)/\partial \theta + f] \odot \dot{\theta}$, where \odot represents the product in the sense of Stratonovich. Then, we can write the entropy productions for each model as

$$\sigma_1 := \frac{Q_1^h}{T_h} + \frac{Q_1^c}{T_c}, \quad (4)$$

$$\sigma_2 := \frac{1}{T_{\text{eff}}(\theta)} \odot Q_2, \quad (5)$$

$$\sigma_3 := \frac{1}{T_{\text{eff}}(\theta)} \odot Q_3. \quad (6)$$

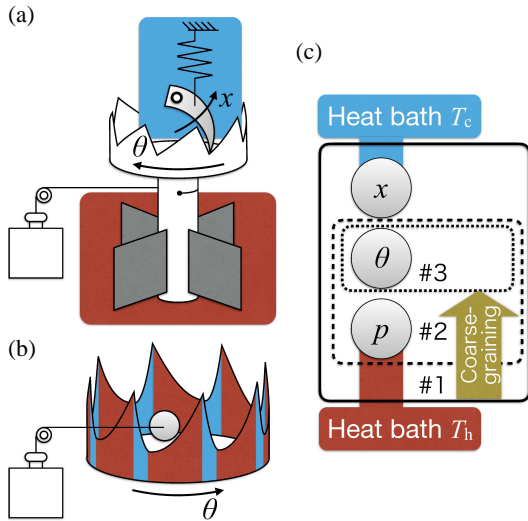


FIG. 1. Schematics of the FS ratchet and its coarse-grained description. (a) In Model-1, the FS ratchet consists of a vane, a gear and a pawl. A spring presses the pawl against the gear, and an external load applies torque to the axle. The vane and pawl are attached to different heat baths. (b) In Models-2 and 3, Langevin equations with the effective mechanical potential, $U_{\text{eff}}(\theta)$, inhomogeneous friction, $\mathcal{G}(\theta)$, and temperature, $T_{\text{eff}}(\theta)$, describe the dynamics of the FS ratchet. (c) The scheme of coarse-graining and the degrees of freedom in each model.

In contrast to $Q_1^h + Q_1^c$ and Q_2 which are equal to the rate of change in the kinetic energy and the (effective) potential energy, Q_3 cannot be interpreted as the change in the system's energy. Thus, it is already clear that σ_3 is different to the conventional entropy production, and is only an effective quantity.

Although irreversibility in nonequilibrium steady states is characterized by the ensemble averages of the entropy production rates, we find in the FS ratchet model that there are finite differences in the averaged entropy production between different models even in the coarse-grained limit, which we call the hidden entropy production. First, from the formula for $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$ derived in [25], we obtain the explicit form for the hidden entropy production rate between the Models-1 and 2:

$$\langle\sigma_1\rangle - \langle\sigma_2\rangle \xrightarrow{\lambda/\gamma \rightarrow \infty} \left\langle \frac{\Gamma(\mathcal{G}(\theta) - \Gamma)}{\mathcal{G}(\theta)T_{\text{eff}}(\theta)} \left(\frac{1}{T_c} - \frac{1}{T_h} \right) (T_h - T_c) \frac{p^2}{m^2} \right\rangle. \quad (7)$$

The hidden entropy production rate is positive when $T_h \neq T_c$, which indicates that the coarse-graining by the tightly confined limit makes the dissipation seemingly decrease.

Next, we find the hidden entropy production between Models-2 and 3, which generalizes the previous results obtained in the coarse-graining limit of the BL motor[15]. Following a similar method [25], we obtain in the limit of $m/\Gamma \rightarrow 0$:

$$\langle\sigma_2\rangle - \langle\sigma_3\rangle \xrightarrow{\lambda/\gamma \rightarrow \infty, m/\Gamma \rightarrow 0} \left\langle \frac{T_{\text{eff}}(\theta)}{2\mathcal{G}(\theta)} \left(\frac{T'_{\text{eff}}(\theta)}{T_{\text{eff}}(\theta)} \right)^2 \right\rangle, \quad (8)$$

which is again positive unless $T_{\text{eff}}(\theta) = \text{const}$. The condition for the positivity of hidden entropy production has been discussed in [16]. The explicit forms of the two hidden entropy production rates Eqs (7,8) are the second main results.

Numerical Simulation.— We performed numerical simulations of the FS ratchet with $\phi(\theta) = \sin(2\pi\theta) + 0.25 \sin(4\pi\theta) + 1.1$, $m = \epsilon$, $\Gamma = 5.0$, $\gamma = 0.05$, $k = 1.0$, $\lambda = \lambda_0/\epsilon$, $T_h = 1.1$, $T_c = 0.9$. Here, ϵ and λ_0 are parameters introduced to control the separation of time scales. The limit of $\lambda_0 \rightarrow \infty$ represents the tight confinement of the pawl to the ratchet, and $\epsilon \rightarrow 0$ realizes the overdamped limit while keeping the ratio of $(\lambda/\gamma)^{-1}$ to m/Γ proportional to λ_0 . The functional forms of $\phi(\theta)$, $U_{\text{eff}}(\theta)$, $\mathcal{G}(\theta)$ and $T_{\text{eff}}(\theta)$ are shown in FIG. 2.

Numerical results of the steady-state entropy production rates are plotted in FIG. 3. In the tightly confined limit which is essentially achieved at $\lambda_0 \gtrsim 10$, the entropy production rate at Model-1, $\langle\sigma_1\rangle$, converges to the sum of $\langle\sigma_2\rangle$ and the analytically obtained first hidden entropy production rate [r.h.s. of Eq. (7)]. Next, fixing the parameter at tightly confined regime, $\lambda_0 = 20$, we see the convergence of $\langle\sigma_2\rangle$ to the sum of $\langle\sigma_3\rangle$ and the second hidden entropy production rate [r.h.s. of Eq. (8)] in the limit of $\epsilon \rightarrow 0$. Furthermore, the entropy production rate at Model-1 $\langle\sigma_1\rangle$ diverges with ϵ^{-1} as shown in FIG. 3. This divergence could be understood by further

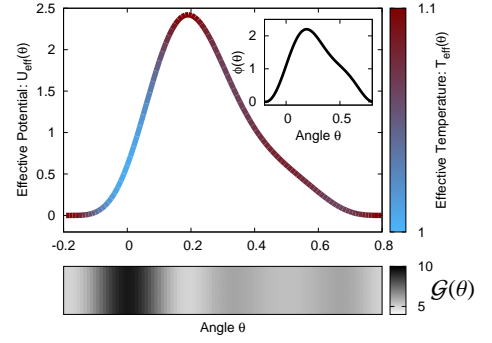


FIG. 2. The functional forms of $\phi(\theta)$, $U_{\text{eff}}(\theta)$, $\mathcal{G}(\theta)$ and $T_{\text{eff}}(\theta)$ we used for the numerical simulation. The temperature difference between the positions of positive and negative potential slopes causes a net flow.

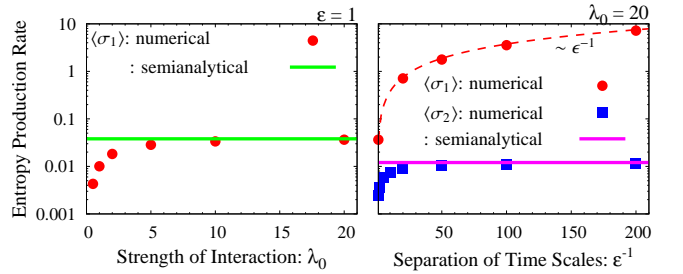


FIG. 3. Steady state entropy production rates. Filled circles and squares (*numerical*) represent the results obtained from the direct numerical calculation. Solid lines (*semianalytical*) are obtained by adding $\langle\sigma_2\rangle$ or $\langle\sigma_3\rangle$ to the numerically evaluated analytical expressions of hidden entropy production rates [Eqs.(7,8)], respectively. The dashed line $\sim \epsilon^{-1}$ shows an asymptotic dependence of $\langle\sigma_1\rangle$ on ϵ .

taking the overdamped limit in Eq. (7):

$$\langle\sigma_1\rangle - \langle\sigma_2\rangle \xrightarrow{\lambda/\gamma \rightarrow \infty, m/\Gamma \rightarrow 0} \left\langle \frac{1}{m} \frac{\Gamma(\mathcal{G}(\theta) - \Gamma)}{\mathcal{G}(\theta)} \left(\frac{1}{T_c} - \frac{1}{T_h} \right) (T_h - T_c) \right\rangle \propto \epsilon^{-1}. \quad (9)$$

To understand better the impact of hidden entropy production, we calculated the thermodynamic efficiencies of the models, η_1 , η_2 and η_3 . In Model-1, we have the natural thermodynamic efficiency $\eta_1 := 1 - \langle Q_c \rangle / \langle Q_h \rangle$ (since $T_h > T_c$ in this simulation). For η_2 and η_3 , we adopted a generalized definition of efficiency [28] to account for the non-uniform continuous temperature. In this definition, the average rates of heat release to the heat bath at a given effective temperature, $\langle Q_{2,3}(T_{\text{eff}}) \rangle$, are introduced by averaging the heat flux over the angle θ with the same effective temperature. The averaged heat release and absorption rates, $\langle Q_{2,3}^{\text{rel}} \rangle$ and $\langle Q_{2,3}^{\text{abs}} \rangle$ are defined as the weighted averages of the positive/negative $\langle Q_{2,3}(T_{\text{eff}}) \rangle$, respectively (see [25]). From these we obtain the generalized efficiencies $\eta_{2,3} := 1 - \langle Q_{2,3}^{\text{rel}} \rangle / \langle Q_{2,3}^{\text{abs}} \rangle$.

We show the result of the efficiencies in FIG. 4. A major difference can be found between $\eta_{1,2}$ and η_3 at $f \rightarrow f_{\text{stall}}$, where f_{stall} is the force that sets the stalled state, $\langle \dot{\theta} \rangle = 0$. The

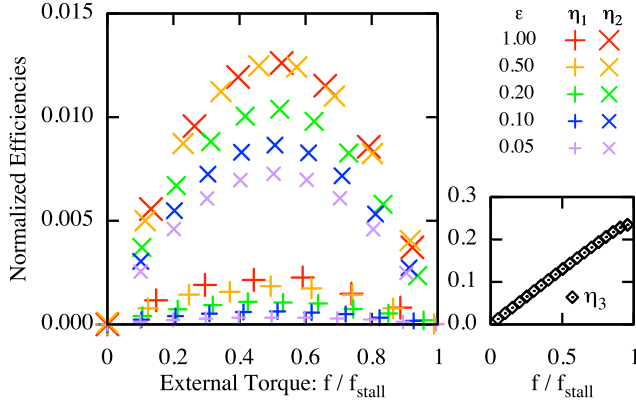


FIG. 4. Thermodynamic efficiencies. The dependence on the external torque, f and the separation of time scale, ϵ , (for η_1 and η_2) are shown. We set the symbols smaller, as ϵ becomes smaller. The efficiencies are normalized by the Carnot efficiency of Model-1, $1 - T_c/T_h$.

non-vanishing η_3 indicates that there is no heat flow from the hot heat bath to cold upon zero work, which corresponds to $\langle\sigma_3\rangle = 0$. In contrast, η_1 and η_2 vanish at $f \rightarrow f_{\text{stall}}$, even in the limit of $\epsilon \rightarrow 0$. This is due to a finite hidden heat flow producing entropy. Moreover, η_1 approaches to 0 regardless of f at $\epsilon \rightarrow 0$ corresponding to the divergence $\langle\sigma_1\rangle \rightarrow \infty$ [Eq. (9)]. These results highlight the effects of coarse-graining on the qualitative behaviors of thermodynamic efficiency; one may assume a significantly higher efficiency of an engine by neglecting the dissipative contributions of the fast variables.

Decomposition of Langevin Dynamics.— Here, we consider if it is possible to reconstruct the thermodynamic irreversibility defined at the fine-grained description from the observation at the coarse-grained scale. In a system where the time scales of variables are well-separated, it is challenging to probe the dynamics of the fast variable, meaning that the hidden entropy production and the real thermodynamic efficiency are almost impossible to measure [29]. Although there is no general workaround to the problem of inaccessible fast variables, we find in Model-2 of the FS ratchet that by using the notion of two heat baths we may split the coarse-grained dynamics and evaluate the entropy production at the fine-grained scale (Model-1). This is obtained by considering the dynamics as a mixture of two Langevin dynamics with different temperatures and frictions corresponding to the two heat baths:

$$\begin{aligned} \dot{\theta} &= \frac{p}{m}, \\ \dot{p} &= -2\frac{\Gamma_i(\theta)}{m}p + \left(f - \frac{\partial U_{\text{eff}}(\theta)}{\partial \theta}\right) + \sqrt{4\Gamma_i(\theta)T_i}\Xi, \end{aligned} \quad (\text{Model-4})$$

instead of a single set of effective temperature and friction [Eqs. (1,2)]. Here, $\Gamma_h(\theta) = \Gamma$, $\Gamma_c(\theta) = \gamma\phi(\theta)^2$ and we consider that i is switched between h and c with a sufficiently fast switching rate Λ . This stochastic process not only reproduces

the dynamics of Model-2 at the sufficiently longer time scale than Λ^{-1} [25], but also gives the entropy production rate, $\langle\sigma_4\rangle$, that satisfies [25]

$$\lim_{\lambda/\gamma \rightarrow \infty} \langle\sigma_1\rangle = \langle\sigma_4\rangle. \quad (10)$$

Equation (10) is useful when we know the original temperatures of the heat baths but can only observe the dynamics at the coarse-grained scale. Since $\mathcal{G}(\theta)$ and $T_{\text{eff}}(\theta)$ can be measured at the coarse-grained scale, we may solve Eqs. (1, 2) using T_h and T_c to obtain $\Gamma_i(\theta)$ in such situation, which allows the evaluation of $\langle\sigma_4\rangle$ as opposed to $\langle\sigma_2\rangle$ which misses out on a finite dissipation [Eq. (7)]. The formulation of entropy production based on the decomposition of the stochastic transition into contributions from different environments has been previously discussed [1, 20, 30]. The approach here is a natural extension of these strategies to the case of a continuous-variable heat engine.

Discussion and conclusion.— Based on the explicit forms of hidden entropy productions and the calculated efficiencies, we here revisit the original discussions by Feynman [19], Parrondo, and Español [20]. As pointed out by Parrondo and Español, Feynman's claim that Carnot efficiency is achievable is equivalent to assuming that the ratchet may work reversibly. In our framework, the time-reversible situation arises at the stalled state in the overdamped scale, where we have $\langle\sigma_3\rangle = 0$. The coarse-grained model matches with Feynman's viewpoint, as he considered the discretized transitions of the ratchets motion excluding the momentum to phenomenologically obtain the thermodynamic efficiency. Parrondo and Español claimed that Feynman's argument failed to count the additional entropy produced in a fine-grained description of the ratchet model. Indeed, if we set $U(\theta, x) = \lambda(\theta - x)^2/2$ and $f = 0$, their model may be considered as an underdamped version of Model-1, and the additional entropy production in their model is equivalent with the hidden entropy production between Model-1 and Model-2.

In this Letter, we clarified how stochastic thermodynamics combined with explicit coarse-graining can shed new light on the classical problem of the FS ratchet. Through deriving the forms of entropy production and numerically obtaining the efficiencies of the FS ratchet, we highlighted how the thermodynamic aspect of a non-equilibrium system can appear different upon coarse-graining. Such difference is crucial not only because focusing on a small number of variables is a natural theoretical procedure as taken by Feynman [19], but also because the measurement in experiment is typically restricted to a small set of slow variables. We believe that the explicit approach on the fine-grained theoretical models will prove useful in the understanding of the efficiencies of nano-scale energy transducers such as bio-molecular motors, since separation of time-scale is a typical issue. In this context, the decomposition of stochastic transitions points to a general workaround to connect stochastic thermodynamics between the different scales.

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Supplemental Material for *Thermodynamic Efficiency of Feynman-Smoluchowski Ratchet: Unified View through Explicit Coarse-Graining*

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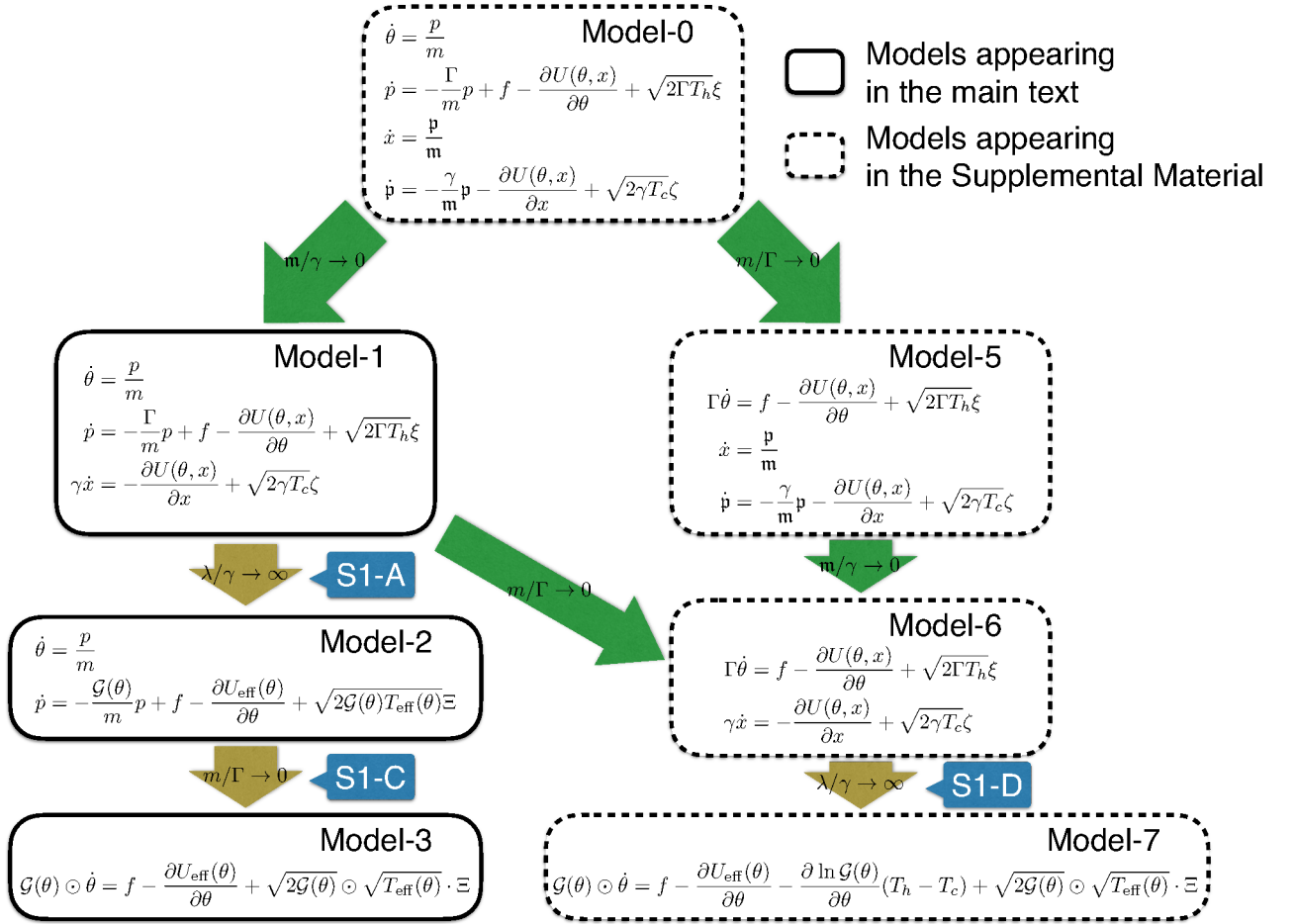


FIG. S1. Complete picture of the series of coarse-graining. Models enclosed in the solid boxes (Model-1,2,3) appear in the main text. Other models (Model-0,5,6,7) appear only in the Supplemental Material. The arrows represent the processes of coarse-graining. The balloons beside the yellow arrows indicate the subsections where we discuss the coarse-graining procedure.

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S1. DERIVATION OF COARSE-GRAINED DYNAMICS

In the main text, we start from Model-1 where θ represents the angle of the coaxial vane and gear, p is the angular momentum conjugated to θ , m is the corresponding moment of inertia, x represents the height of the pawl, f is the torque exerted on the axle, and the two Langevin heat baths are described by independent white Gaussian noises with zero means and unit variances, ξ and ζ , together with the temperatures, T_h and T_c , and the viscous frictional coefficients, Γ and γ . Model-2 and Model-3 are obtained by taking the tightly confined limit and the overdamped limit successively. The calculations describing the two coarse-graining are discussed in the following subsections A and C.

In FIG. S1, a complete picture of a series of coarse-graining is shown. Model-1 has an underdamped version, which we call Model-0 where p and m are the momentum and the mass of pawl, respectively. Starting from Model-0, it is possible to take the overdamped limit for the vane and gear first, which results in Model-5. Either through Model-1 or 5, we can obtain Model-6, which is the overdamped model of the Feynman-Smoluchowski ratchet. Note that the coarse-graining of momentum degrees of freedom attached to the homogeneous heat bath (green arrows in FIG. S1) is trivial and does not involve hidden entropy production. Finally, by taking the tightly-confined limit in Model-6, we obtain a single-variable overdamped dynamics, Model-7 (subsection D). The discrepancy between Model-3 and 7, arises due to the difference in order of the two limits.

A. Derivation of Model-2 based on Singular Perturbation Theory

In this subsection, we derive the coarse-grained dynamics corresponding to Model-2 from a generalized version of Model-1. The generalized model differs from the original Model-1 in the choice of $U(\theta, x)$. We here require $U(\theta, x)$ to be decomposed as

$$U(\theta, x) = U_I \left(\frac{x - \phi(\theta)}{\sqrt{\varepsilon}} \right) + U_0(x), \quad (\text{S1})$$

where U_I is a trapping potential function, and ε is a dimensionless small parameter. If we choose $U_I(d) = \lambda_0 d^2/2$ and $U_0(x) = kx^2/2$, the original Model-1 is reproduced, where $\lambda = \lambda_0/\varepsilon$.

We employ the singular perturbation theory of Kramers-Fokker-Planck equation to derive the coarse-grained dynamics. The Kramers-Fokker-Planck equation corresponding to Model-1, may be written as

$$\begin{aligned} \frac{\partial P(\theta, p, x)}{\partial t} = & - \frac{\partial}{\partial \theta} \left(\frac{p}{m} P(\theta, p, x) \right) - \frac{\partial}{\partial p} \left[\left(-\frac{\Gamma}{m} p - \frac{\partial U(\theta, x)}{\partial \theta} + f \right) P(\theta, p, x) - \Gamma T_h \frac{\partial P(\theta, p, x)}{\partial p} \right] \\ & - \frac{\partial}{\partial x} \left(-\frac{1}{\gamma} \frac{\partial U(\theta, x)}{\partial x} P(\theta, p, x) - \frac{T_c}{\gamma} \frac{\partial P(\theta, p, x)}{\partial x} \right), \end{aligned} \quad (\text{S2})$$

where $P(\theta, p, x)$ is the joint probability density of θ , p , and x . Our goal here is to obtain the coarse-grained dynamics, that is, the differential equation for the joint probability density of θ and p , $P(\theta, p)$, which is obtained by integrating out x from $P(\theta, p, x)$:

$$P(\theta, p) = \int dx P(\theta, p, x). \quad (\text{S3})$$

Throughout this Supplemental Material, the integrals with respect to θ, p, x , and rescaled variables are taken over the domain of integrand. Although the time derivative of $P(\theta, p)$ is formally written as

$$\frac{\partial P(\theta, p)}{\partial t} = - \frac{\partial}{\partial \theta} \left(\frac{p}{m} P(\theta, p) \right) - \frac{\partial}{\partial p} \left[\left(-\frac{\Gamma}{m} p + f \right) P(\theta, p) - \int dx \frac{\partial U(\theta, x)}{\partial \theta} P(\theta, p, x) - \Gamma T_h \frac{\partial P(\theta, p)}{\partial p} \right], \quad (\text{S4})$$

this does not give the coarse-grained dynamics, since the right hand side of Eq. (S4) still depends on $P(\theta, p, x)$. The heart of the singular perturbation theory is to decompose the time-dependence of $P(\theta, p, x)$ into the explicit part and the implicit part through $P(\theta, p)$. In an appropriate limit, the explicit part decays quickly and the right hand side of Eq. (S4) essentially turns into a functional of $P(\theta, p)$.

The problem of the singular perturbation theory is mapped onto that of the ordinary perturbation theory by introducing M and Ω which describe $P(\theta, p, x)$ and the dynamics of $P(\theta, p)$, respectively. For this purpose, we first switch the variables from (t, x) to the scaled time and distance

$$\mathcal{T} := \varepsilon^{-1} t, \quad s := \frac{x - \phi(\theta)}{\sqrt{\varepsilon}}, \quad (\text{S5})$$

as

$$\begin{aligned} \varepsilon^{-1} \frac{\partial P(\theta, p, s)}{\partial \mathcal{T}} = & -\frac{\partial}{\partial \theta} \left(\frac{p}{m} P(\theta, p, s) \right) + \phi'(\theta) \varepsilon^{-1/2} \frac{\partial}{\partial s} \left(\frac{p}{m} P(\theta, p, s) \right) \\ & - \frac{\partial}{\partial p} \left[\left(-\frac{\Gamma}{m} p + \varepsilon^{-1/2} \phi'(\theta) \frac{\partial U_I(s)}{\partial s} + f \right) P(\theta, p, s) - \Gamma T_h \frac{\partial P(\theta, p, s)}{\partial p} \right] \\ & - \varepsilon^{-1} \frac{\partial}{\partial s} \left(-\frac{1}{\gamma} \left(\frac{\partial U_I(s)}{\partial s} + \varepsilon^{1/2} U'_0(\phi(\theta)) + O(\varepsilon) \right) P(\theta, p, s) - \frac{T_c}{\gamma} \frac{\partial P(\theta, p, s)}{\partial s} \right). \end{aligned} \quad (\text{S6})$$

The explicit and implicit dependence of $P(\theta, p, s)$ on \mathcal{T} is implemented by describing $P(\theta, p, s)$ as an output of a \mathcal{T} -dependent operator, M , on $P(\theta, p)$:

$$P(\theta, p, s) = M[P(\theta', p'); \mathcal{T}](\theta, p, s). \quad (\text{S7})$$

Furthermore, we represent the time-evolution of $P(\theta, p)$ by a \mathcal{T} -dependent operator, Ω , on $P(\theta, p)$:

$$\begin{aligned} \frac{\partial P(\theta, p)}{\partial \mathcal{T}} = \Omega[P(\theta', p'); \mathcal{T}](\theta, p) := & -\varepsilon \frac{\partial}{\partial \theta} \left(\frac{p}{m} P(\theta, p) \right) - \varepsilon \frac{\partial}{\partial p} \left[\left(-\frac{\Gamma}{m} p + f \right) P(\theta, p) - \Gamma T_h \frac{\partial P(\theta, p)}{\partial p} \right] \\ & - \frac{\partial}{\partial p} \left[\varepsilon^{1/2} \phi'(\theta) \int ds \frac{\partial U_I(s)}{\partial s} M[P(\theta', p'); \mathcal{T}](\theta, p, s) \right]. \end{aligned} \quad (\text{S8})$$

Since M depends on \mathcal{T} explicitly and implicitly [through $P(\theta, p)$], the substitution of M into the left hand side of Eq. (S6) gives

$$\begin{aligned} [\text{l.h.s. of Eq. (S6)}] = & \frac{\partial M[P(\theta', p'); \mathcal{T}](\theta, p, s)}{\partial \mathcal{T}} + \int d\theta'' dp'' \frac{\partial P(\theta'', p'')}{\partial \mathcal{T}} \frac{\delta M[P(\theta'', p''); \mathcal{T}](\theta, p, s)}{\delta P(\theta'', p'')} \\ = & \frac{\partial M[P(\theta', p'); \mathcal{T}](\theta, p, s)}{\partial \mathcal{T}} + \int d\theta'' dp'' \Omega[P(\theta', p'); \mathcal{T}](\theta'', p'') \frac{\delta M[P(\theta'', p''); \mathcal{T}](\theta, p, s)}{\delta P(\theta'', p'')} \end{aligned} \quad (\text{S9})$$

according to the chain rule. Applying Eq. (S7) also on the right hand side of Eq. (S6), we obtain

$$\begin{aligned} & \frac{\partial M[P(\theta', p'); \mathcal{T}](\theta, p, s)}{\partial \mathcal{T}} + \int d\theta'' dp'' \Omega[P(\theta', p'); \mathcal{T}](\theta'', p'') \frac{\delta M[P(\theta'', p''); \mathcal{T}](\theta, p, s)}{\delta P(\theta'', p'')} \\ = & -\varepsilon \frac{\partial}{\partial \theta} \left(\frac{p}{m} M[P(\theta', p'); \mathcal{T}](\theta, p, s) \right) + \phi'(\theta) \varepsilon^{1/2} \frac{\partial}{\partial s} \left(\frac{p}{m} M[P(\theta', p'); \mathcal{T}](\theta, p, s) \right) \\ & - \varepsilon \frac{\partial}{\partial p} \left[\left(-\frac{\Gamma}{m} p + \varepsilon^{-1/2} \phi'(\theta) \frac{\partial U_I(s)}{\partial s} + f \right) M[P(\theta', p'); \mathcal{T}](\theta, p, s) - \Gamma T_h \frac{\partial M[P(\theta', p'); \mathcal{T}](\theta, p, s)}{\partial p} \right] \\ & - \frac{\partial}{\partial s} \left[-\frac{1}{\gamma} \left(\frac{\partial U_I(s)}{\partial s} + \varepsilon^{1/2} U'_0(\phi(\theta)) + O(\varepsilon) \right) M[P(\theta', p'); \mathcal{T}](\theta, p, s) - \frac{T_c}{\gamma} \frac{\partial M[P(\theta', p'); \mathcal{T}](\theta, p, s)}{\partial s} \right]. \end{aligned} \quad (\text{S10})$$

The remaining task is to apply the standard procedure of perturbation theory. We expand M and Ω into series of $\varepsilon^{1/2}$:

$$M[P(\theta', p'); \mathcal{T}](\theta, p, s) = \sum_{n=0} \varepsilon^{n/2} M^{(n)}[P(\theta', p'); \mathcal{T}](\theta, p, s), \quad (\text{S11})$$

$$\Omega[P(\theta', p'); \mathcal{T}](\theta, p) = \sum_{n=0} \varepsilon^{(n+1)/2} \Omega^{(n)}[P(\theta', p'); \mathcal{T}](\theta, p). \quad (\text{S12})$$

Here, the difference in the lowest order for M and Ω is due to Eqs. (S8). The leading order of Eq. (S10) gives

$$\frac{\partial M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s)}{\partial \mathcal{T}} = -\frac{\partial}{\partial s} \left[-\frac{1}{\gamma} \frac{\partial U_I(s)}{\partial s} M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s) - \frac{T_c}{\gamma} \frac{\partial M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s)}{\partial s} \right], \quad (\text{S13})$$

from which we obtain

$$M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s) = P(\theta, p) \frac{\exp(-U_I(s)/T_c)}{Z} + [\text{exponentially decaying terms}], \quad (\text{S14})$$

where [exponentially decaying terms] depend on \mathcal{T} explicitly and decay exponentially at the time scale of $O(1)$, and $Z = \int ds \exp(-U_I(s)/T_c)$. Since we have interest in the longer time scale than $O(1)$, we neglect

[exponentially decaying terms] hereafter. Under this assumption of time scale, $\Omega^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p)$ vanishes, since the last term in the right hand side of Eq. (S8) is zero in the leading order. The sub-leading order of Eq. (S10) is

$$\begin{aligned} \frac{\partial M^{(1)}[P(\theta', p'); \mathcal{T}](\theta, p, s)}{\partial \mathcal{T}} &= \phi'(\theta) \frac{\partial}{\partial s} \left(\frac{p}{m} M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s) \right) - \frac{\partial}{\partial p} \left(\phi'(\theta) \frac{\partial U_I(s)}{\partial s} M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s) \right) \\ &\quad - \frac{1}{\gamma} \frac{\partial}{\partial s} \left(-\frac{\partial U_I(s)}{\partial s} M^{(1)}[P(\theta', p'); \mathcal{T}](\theta, p, s) - U'_0(\phi(\theta)) M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s) \right. \\ &\quad \left. - T_c \frac{\partial M^{(1)}[P(\theta', p'); \mathcal{T}](\theta, p, s)}{\partial s} \right), \end{aligned} \quad (\text{S15})$$

which has a particular solution

$$\begin{aligned} M^{(1)}[P(\theta', p'); \mathcal{T}](\theta, p, s) &= - \left\{ T_c \phi'(\theta) \frac{\partial P(\theta, p)}{\partial p} + \left[\phi'(\theta) \frac{p}{m} + \frac{U'_0(\phi(\theta))}{\gamma} \right] P(\theta, p) \right\} \frac{\gamma}{T_c} s \frac{\exp(-U_I(s)/T_c)}{Z} \\ &\quad + [\text{exponentially decaying terms}]. \end{aligned} \quad (\text{S16})$$

By substituting Eq. (S16) into Eq. (S8),

$$\begin{aligned} \Omega^{(1)}[P(\theta', p'); \mathcal{T}] &= - \frac{\partial}{\partial \theta} \left(\frac{p}{m} P(\theta, p) \right) - \frac{\partial}{\partial p} \left[\left(-\frac{\Gamma}{m} p + f \right) P(\theta, p) - \Gamma T_h \frac{\partial P(\theta, p)}{\partial p} \right] \\ &\quad - \frac{\partial}{\partial p} \left[\phi'(\theta) \int ds \frac{\partial U_I(s)}{\partial s} M^{(1)}[P(\theta', p'); \mathcal{T}](\theta, p, s) \right] \\ &= - \frac{\partial}{\partial \theta} \left(\frac{p}{m} P(\theta, p) \right) - \frac{\partial}{\partial p} \left[\left(-\frac{\Gamma}{m} p + f \right) P(\theta, p) - \Gamma T_h \frac{\partial P(\theta, p)}{\partial p} \right] \\ &\quad - \frac{\partial}{\partial p} \left\{ -\gamma \phi'(\theta) \left[T_c \phi'(\theta) \frac{\partial P(\theta, p)}{\partial p} + \left(\phi'(\theta) \frac{p}{m} + \frac{U'_0(\phi(\theta))}{\gamma} \right) P(\theta, p) \right] \right\}. \end{aligned} \quad (\text{S17})$$

The Kramers equation immediately follows from the relation, $\partial P(\theta, p)/\partial \mathcal{T} = \Omega[P(\theta', p'); \mathcal{T}]$:

$$\frac{\partial P(\theta, p)}{\partial t} = - \frac{\partial}{\partial \theta} \left(\frac{p}{m} P(\theta, p) \right) - \frac{\partial}{\partial p} \left[\left(-\frac{\Gamma + \gamma \phi'(\theta)^2}{m} p - \frac{\partial U_0(\phi(\theta))}{\partial \theta} + f \right) P(\theta, p) - (\Gamma T_h + \gamma \phi'(\theta)^2 T_c) \frac{\partial P(\theta, p)}{\partial p} \right], \quad (\text{S18})$$

which corresponds to the underdamped Langevin equation of Model-2 with

$$U_{\text{eff}}(\theta) = U_0(\phi(\theta)), \quad (\text{S19})$$

$$\mathcal{G}(\theta) = \Gamma + \gamma \phi'(\theta)^2, \quad (\text{S20})$$

$$T_{\text{eff}}(\theta) = \frac{\Gamma T_h + \gamma \phi'(\theta)^2 T_c}{\Gamma + \gamma \phi'(\theta)^2}. \quad (\text{S21})$$

B. Quick Derivation of Model-2 for Harmonic Potential

In this subsection, we give an alternative and quick derivation of the coarse-grained dynamics for the case of harmonic potential [Eq. (1) in the main text], based on the approach of Zwanzig [R. Zwanzig, *J. Stat. Phys.* **9**, 215 (1973)]. In this derivation, we formally solve the equation of motion of the pawl, and substitute it into that of the gear and the vane. The equation of motion of the pawl,

$$\gamma \dot{x}(t) = - \frac{\partial U(\theta(t), x(t))}{\partial x(t)} + \sqrt{2\gamma T_c} \zeta(t) = -\lambda[x(t) - \phi(\theta(t))] - kx(t) + \sqrt{2\gamma T_c} \zeta(t) \quad (\text{S22})$$

is solved as

$$\begin{aligned} x(t) &= \frac{1}{\gamma} \int_{-\infty}^t dt' \exp\left(-\frac{\lambda+k}{\gamma}(t-t')\right) [\lambda\phi(\theta(t')) + \sqrt{2\gamma T_c} \zeta(t')] \\ &= \frac{\lambda}{\lambda+k} \phi(\theta(t)) - \int_{-\infty}^t dt' \exp\left(-\frac{\lambda+k}{\gamma}(t-t')\right) \left[\frac{\lambda}{\lambda+k} \dot{\theta}(t') \phi'(\theta(t')) + \frac{\sqrt{2\gamma T_c}}{\gamma} \zeta(t') \right], \end{aligned} \quad (\text{S23})$$

where we explicitly show the time-dependence of x , θ and ζ . By substituting Eq. (S23) into the equation of motion of the gear and the vane, we obtain

$$\begin{aligned}\dot{\theta}(t) &= \frac{p(t)}{m} \\ \dot{p}(t) &= -\frac{\Gamma}{m}p(t) + f + \sqrt{2\Gamma T_h}\xi(t) - \frac{k\phi(\theta(t))}{\lambda+k}\lambda\phi'(\theta(t)) \\ &\quad - \lambda\phi'(\theta(t)) \int_{-\infty}^t dt' \exp\left(-\frac{\lambda+k}{\gamma}(t-t')\right) \left[\frac{\lambda}{\lambda+k}\dot{\theta}(t')\phi'(\theta(t')) + \frac{\sqrt{2\gamma T_c}}{\gamma}\zeta(t') \right].\end{aligned}\quad (\text{S24})$$

Since, in the limit of $\lambda/\gamma \rightarrow \infty$,

$$\int_{-\infty}^t dt' \exp\left(-\frac{\lambda+k}{\gamma}(t-t')\right) \varphi(t') \rightarrow \frac{\gamma}{\lambda+k} \varphi(t) \quad (\text{S25})$$

holds for an arbitrary function φ , we finally reach the underdamped Langevin equation corresponding to Model-2,

$$\begin{aligned}\dot{\theta}(t) &= \frac{p(t)}{m} \\ \dot{p}(t) &= -\frac{\Gamma}{m}p(t) + f + \sqrt{2\Gamma T_h}\xi(t) - k\phi'(\theta(t))\phi(\theta(t)) - \left[\gamma\dot{\theta}(t)\phi'(\theta(t))^2 + \sqrt{2\gamma\phi'(\theta(t))^2 T_c}\zeta(t) \right] \\ &= -\frac{\mathcal{G}(\theta(t))}{m}p(t) + f - k\phi'(\theta(t))\phi(\theta(t)) + \sqrt{2\mathcal{G}(\theta(t))T_{\text{eff}}(\theta(t))}\Xi(t).\end{aligned}\quad (\text{S26})$$

Although this derivation is much simpler than the general method presented in the previous subsection, it does not explicitly give the asymptotic form of the probability density, which is essential in evaluating the entropy production rate [see §S2].

C. Derivation of Overdamped Dynamics

The coarse-graining from Model-2 to Model-3 may be also formulated in the framework of §S1-A. The Kramers equation corresponding to Model-2 [Eq. (S18)] may be rewritten as

$$\frac{\partial P(\theta, p)}{\partial t} = -\frac{\partial}{\partial \theta} \left(\frac{p}{m} P(\theta, p) \right) - \frac{\partial}{\partial p} \left[\left(-\frac{\mathcal{G}(\theta)}{m} p - \frac{\partial U_0(\phi(\theta))}{\partial \theta} + f \right) P(\theta, p) - \mathcal{G}(\theta) T_{\text{eff}}(\theta) \frac{\partial P(\theta, p)}{\partial p} \right], \quad (\text{S27})$$

in terms of the effective friction coefficient and temperature. By introducing the scaled time and momentum

$$\tau = \frac{\Gamma}{m}t, \quad \tilde{p} = \sqrt{\frac{m_0}{m}}p, \quad (\text{S28})$$

where m_0 is the reference point of mass, we obtain an equation to which we apply the singular perturbation theory,

$$\begin{aligned}\frac{\Gamma}{m} \frac{\partial P(\theta, \tilde{p})}{\partial \tau} &= -\frac{\partial}{\partial \theta} \left(\frac{\tilde{p}}{\sqrt{mm_0}} P(\theta, \tilde{p}) \right) - \frac{\partial}{\partial \tilde{p}} \left[\left(-\frac{\Gamma}{m} \frac{\mathcal{G}(\theta)}{\Gamma} \tilde{p} - \sqrt{\frac{m_0}{m}} \frac{\partial U_0(\phi(\theta))}{\partial \theta} + \sqrt{\frac{m_0}{m}} f \right) P(\theta, \tilde{p}) \right. \\ &\quad \left. - \frac{\Gamma}{m} \frac{m_0 \mathcal{G}(\theta) T_{\text{eff}}(\theta)}{\Gamma} \frac{\partial P(\theta, \tilde{p})}{\partial \tilde{p}} \right].\end{aligned}\quad (\text{S29})$$

Here, m/Γ acts as a small parameter. Following the procedure in §S1-A, we define

$$\tilde{M}[P(\theta'); \tau](\theta, \tilde{p}) := P(\theta, \tilde{p}) \quad (\text{S30})$$

$$\tilde{\Omega}[P(\theta'); \tau](\theta) := -\frac{m}{\Gamma} \int d\tilde{p} \frac{\partial}{\partial \theta} \left(\frac{\tilde{p}}{\sqrt{mm_0}} \tilde{M}[P(\theta'); \tau](\theta, \tilde{p}) \right) \quad (\text{S31})$$

$$= \int d\tilde{p} \frac{\partial P(\theta, \tilde{p})}{\partial \tau} = \frac{\partial P(\theta)}{\partial \tau} \quad (\text{S32})$$

and apply the standard perturbation theory to Eq. (S29) expressed in terms of \tilde{M} and $\tilde{\Omega}$. The leading order gives,

$$\frac{\partial \tilde{M}^{(0)}[P(\theta'); \tau](\theta, \tilde{p})}{\partial \tau} = -\frac{\partial}{\partial \tilde{p}} \left[-\frac{\mathcal{G}(\theta)}{\Gamma} \tilde{p} \tilde{M}^{(0)}[P(\theta'); \tau](\theta, \tilde{p}) - \frac{m_0 \mathcal{G}(\theta) T_{\text{eff}}(\theta)}{\Gamma} \frac{\partial \tilde{M}^{(0)}[P(\theta'); \tau](\theta, \tilde{p})}{\partial \tilde{p}} \right]. \quad (\text{S33})$$

which has a solution

$$\tilde{M}^{(0)}[P(\theta'); \tau](\theta, \tilde{p}) = P(\theta) \frac{\exp\left(-\frac{\tilde{p}^2}{2m_0 T(\theta)}\right)}{\sqrt{2\pi T(\theta)}} + [\text{exponentially decaying terms}]. \quad (\text{S34})$$

Since $\tilde{\Omega}^{(0)}[P(\theta'); \tau](\theta)$ again vanishes, we should proceed to the sub-leading order of Eq. (S29),

$$\begin{aligned} \frac{\partial \tilde{M}^{(1)}[P(\theta'); \tau](\theta, \tilde{p})}{\partial \tau} = & -\frac{\partial}{\partial \theta} \left(\frac{\tilde{p}}{\sqrt{m m_0}} \tilde{M}^{(0)}[P(\theta'); \tau](\theta, \tilde{p}) \right) \\ & -\frac{\partial}{\partial \tilde{p}} \left[\sqrt{\frac{m_0}{m}} \left(-\frac{\partial U_0(\phi(\theta))}{\partial \theta} + f \right) \tilde{M}^{(0)}[P(\theta'); \tau](\theta, \tilde{p}) \right] \\ & -\frac{\partial}{\partial \tilde{p}} \left[-\frac{\mathcal{G}(\theta)}{\Gamma} \tilde{p} \tilde{M}^{(1)}[P(\theta'); \tau](\theta, \tilde{p}) - \frac{m_0 \mathcal{G}(\theta) T_{\text{eff}}(\theta)}{\Gamma} \frac{\partial \tilde{M}^{(1)}[P(\theta'); \tau](\theta, \tilde{p})}{\partial \tilde{p}} \right], \end{aligned} \quad (\text{S35})$$

which has a particular solution

$$\begin{aligned} \tilde{M}^{(1)}[P(\theta'); \tau](\theta, \tilde{p}) = & \left\{ -\frac{\partial P(\theta)}{\partial \theta} - \left[\left(\frac{\tilde{p}^2}{6m_0 T_{\text{eff}}(\theta)} + \frac{1}{2} \right) \frac{T'_{\text{eff}}(\theta)}{T_{\text{eff}}(\theta)} + \frac{1}{T_{\text{eff}}(\theta)} \left(\frac{\partial U_0(\phi(\theta))}{\partial \theta} - f \right) \right] P(\theta) \right\} \\ & \cdot \sqrt{\frac{m}{m_0}} \frac{\Gamma}{m \mathcal{G}(\theta)} \tilde{p} \frac{\exp\left(-\tilde{p}^2/2m_0 T(\theta)\right)}{\sqrt{2\pi T(\theta)}}. \end{aligned} \quad (\text{S36})$$

By substituting Eq. (S36) into Eq. (S31),

$$\begin{aligned} \tilde{\Omega}^{(1)}[P(\theta'); \tau](\theta) := & -\frac{m}{\Gamma} \int d\tilde{p} \frac{\partial}{\partial \theta} \left(\frac{\tilde{p}}{\sqrt{m m_0}} \tilde{M}^{(1)}[P(\theta'); \tau](\theta, \tilde{p}) \right) \\ = & -\frac{\partial}{\partial \theta} \left\{ \frac{1}{\mathcal{G}(\theta)} \left[-\left(\frac{\partial U_0(\phi(\theta))}{\partial \theta} - f + \frac{\partial T_{\text{eff}}(\theta)}{\partial \theta} \right) P(\theta) - T_{\text{eff}}(\theta) \frac{\partial P(\theta)}{\partial \theta} \right] \right\}. \end{aligned} \quad (\text{S37})$$

We finally reach

$$\begin{aligned} \frac{\partial P(\theta)}{\partial t} = & -\frac{\partial}{\partial \theta} \left\{ \frac{1}{\mathcal{G}(\theta)} \left[-\left(\frac{\partial U_0(\phi(\theta))}{\partial \theta} - f + \frac{\partial T_{\text{eff}}(\theta)}{\partial \theta} \right) P(\theta) - T_{\text{eff}}(\theta) \frac{\partial P(\theta)}{\partial \theta} \right] \right\} \\ = & -\frac{\partial}{\partial \theta} \left\{ -\frac{1}{\mathcal{G}(\theta)} \left(\frac{\partial U_0(\phi(\theta))}{\partial \theta} - f + \frac{1}{2\mathcal{G}(\theta)} \frac{\partial}{\partial \theta} [\mathcal{G}(\theta) T_{\text{eff}}(\theta)] \right) P(\theta) - \sqrt{\frac{T_{\text{eff}}(\theta)}{\mathcal{G}(\theta)}} \frac{\partial}{\partial \theta} \left(\sqrt{\frac{T_{\text{eff}}(\theta)}{\mathcal{G}(\theta)}} P(\theta) \right) \right\}. \end{aligned} \quad (\text{S38})$$

In order to obtain the corresponding overdamped Langevin equation, we make use of the correspondence between the Fokker-Planck and Langevin descriptions [N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. (Elsevier, 2007)]. A multiplicative Langevin equation with the product in the sense of Stratonovich

$$\dot{X} = A(X) + C(X) \circ \Xi, \quad (\text{S39})$$

can be mapped to an additive Langevin equation

$$\dot{\bar{X}} = \bar{A}(\bar{X}) + \Xi, \quad (\text{S40})$$

where Ξ is white Gaussian noise with zero mean and unit variance, and $\bar{X} = \int^X dX/C(X)$, $\bar{A}(\bar{X}) = A(X)/C(X)$. Since Eq. (S40) trivially corresponds to the Fokker-Planck equation,

$$\frac{\partial P(\bar{X})}{\partial t} = -\frac{\partial}{\partial \bar{X}} \left(\bar{A}(\bar{X}) P(\bar{X}) - \frac{1}{2} \frac{\partial P(\bar{X})}{\partial \bar{X}} \right) \quad (\text{S41})$$

we obtain the Fokker-Planck equation for $P(X)$ through variable transformation [note that $P(\bar{X}) = P(X)C(X)$]:

$$\frac{\partial P(X)}{\partial t} = -\frac{\partial}{\partial X} \left(A(X)P(X) - \frac{C(X)}{2} \frac{\partial}{\partial X} [C(X)P(X)] \right). \quad (\text{S42})$$

Therefore, the Langevin equation of Model-3 is described as

$$\dot{\theta} = \frac{1}{\mathcal{G}(\theta)} \left(-\frac{\partial U_0(\phi(\theta))}{\partial \theta} + f \right) - \frac{1}{2\mathcal{G}(\theta)} \frac{\partial}{\partial \theta} [\mathcal{G}(\theta)T_{\text{eff}}(\theta)] + \sqrt{2\frac{T_{\text{eff}}(\theta)}{\mathcal{G}(\theta)}} \circ \Xi, \quad (\text{S43})$$

which may be rewritten by using the conversion formula among the Itô, Stratonovich and anti-Itô products as

$$\dot{\theta} = \frac{1}{\mathcal{G}(\theta)} \left(-\frac{\partial U_0(\phi(\theta))}{\partial \theta} + f \right) + \sqrt{\frac{2}{\mathcal{G}(\theta)}} \odot \sqrt{T_{\text{eff}}(\theta)} \cdot \Xi, \quad (\text{S44})$$

and by multiplying $\mathcal{G}(\theta)$ in the sense of anti-Itô as

$$\mathcal{G}(\theta) \odot \dot{\theta} = \left(-\frac{\partial U_0(\phi(\theta))}{\partial \theta} + f \right) + \sqrt{2\mathcal{G}(\theta)} \odot \sqrt{T_{\text{eff}}(\theta)} \cdot \Xi. \quad (\text{S45})$$

D. Derivation of Another Overdamped Dynamics

We show here that by reversing the order of two limits, tightly confined limit and overdamped limit, we obtain a different overdamped dynamics (Model-7). Since the overdamped limit for the pawl in Model-1 may be taken in the usual manner, we here start from the overdamped description of the Feynman-Smoluchowski ratchet:

$$\begin{aligned} \Gamma \dot{\theta} &= -\frac{\partial U(\theta, x)}{\partial \theta} + f + \sqrt{2\Gamma T_h} \xi, \\ \gamma \dot{x} &= -\frac{\partial U(\theta, x)}{\partial x} + \sqrt{2\Gamma T_c} \zeta, \\ U(\theta, x) &= \frac{\lambda}{2} [x - \phi(\theta)]^2 + \frac{k}{2} x^2. \end{aligned} \quad (\text{Model-6}) \quad (\text{S46})$$

The Fokker-Planck equation corresponding to Model-6 may be written as

$$\begin{aligned} \frac{\partial P(\theta, x)}{\partial t} &= -\frac{\partial}{\partial \theta} \left[\frac{1}{\Gamma} \{ \lambda(x - \phi(\theta))\phi'(\theta) + f \} P(\theta, x) - \frac{T_h}{\Gamma} \frac{\partial P(\theta, x)}{\partial \theta} \right] \\ &\quad - \frac{\partial}{\partial x} \left[-\frac{1}{\gamma} (kx + \lambda(x - \phi(\theta))) P(\theta, x) - \frac{T_c}{\gamma} \frac{\partial P(\theta, x)}{\partial x} \right]. \end{aligned} \quad (\text{S47})$$

In the same way as §S1-A, we introduce the scaled time, distance and interaction strength, $\mathcal{T} = \epsilon^{-1}t$, $s = [x - \phi(\theta)]/\sqrt{\epsilon}$ and $\lambda_0 = \epsilon\lambda$. Transforming the variables from (t, x) to (\mathcal{T}, s) , we may rewrite Eq. (S47) as

$$\begin{aligned} \epsilon^{-1} \frac{\partial P(\theta, s)}{\partial \mathcal{T}} &= -\frac{\partial}{\partial \theta} \left[\frac{f}{\Gamma} P(\theta, s) - \frac{T_h}{\Gamma} \frac{\partial}{\partial \theta} P(\theta, s) \right] \\ &\quad - \epsilon^{-1/2} \frac{\partial}{\partial \theta} \left[\frac{\lambda_0}{\Gamma} s \phi'(\theta) P(\theta, s) + \frac{T_h}{\Gamma} \phi'(\theta) \frac{\partial}{\partial s} P(\theta, s) \right] \\ &\quad + \epsilon^{-1/2} \phi'(\theta) \frac{\partial}{\partial s} \left[\frac{f}{\Gamma} P(\theta, s) - \frac{T_h}{\Gamma} \frac{\partial}{\partial \theta} P(\theta, s) \right] - \epsilon^{-1/2} \frac{\partial}{\partial s} \left[-\frac{k}{\gamma} \phi(\theta) P(\theta, s) \right] \\ &\quad - \epsilon^{-1} \frac{\partial}{\partial s} \left[-\frac{\lambda_0}{\Gamma} s \phi'(\theta)^2 P(\theta, s) - \frac{T_h}{\Gamma} \phi'(\theta)^2 \frac{\partial}{\partial s} P(\theta, s) - \frac{1}{\gamma} \lambda_0 s P(\theta, s) - \frac{T_c}{\gamma} \frac{\partial}{\partial s} P(\theta, s) \right]. \end{aligned} \quad (\text{S48})$$

Again, following the procedure in §S1-A, we define

$$\hat{M}[P(\theta'; t)](\theta, s) := P(\theta, s) \quad (\text{S49})$$

$$\hat{\Omega}[P(\theta; t)] := -\frac{\partial}{\partial \theta} \left[\frac{f}{\Gamma} P(\theta) - \frac{T_h}{\Gamma} \frac{\partial}{\partial \theta} P(\theta) \right] - \epsilon^{-1/2} \frac{\partial}{\partial \theta} \left[\int ds \frac{\lambda_0}{\Gamma} s \phi'(\theta) \hat{M}[P(\theta'; t)](\theta, s) \right]. \quad (\text{S50})$$

Now, we apply the standard perturbation theory to Eq. (S48) expressed in terms of \hat{M} and $\hat{\Omega}$. Since the leading order gives

$$\begin{aligned}\frac{\partial \hat{M}^{(0)}}{\partial \mathcal{T}} &= -\frac{\partial}{\partial s} \left[-\frac{\lambda_0}{\Gamma} s \phi'(\theta)^2 \hat{M}^{(0)} - \frac{T_h}{\Gamma} \phi'(\theta)^2 \frac{\partial}{\partial s} \hat{M}^{(0)} - \frac{1}{\gamma} \lambda_0 s \hat{M}^{(0)} - \frac{T_c}{\gamma} \frac{\partial}{\partial s} \hat{M}^{(0)} \right] \\ &= -\frac{1}{\Gamma \gamma} \frac{\partial}{\partial s} \left[-\lambda_0 s [\Gamma + \gamma \phi'(\theta)^2] \hat{M}^{(0)} - [T_h \gamma \phi'(\theta)^2 + T_c \Gamma] \frac{\partial}{\partial s} \hat{M}^{(0)} \right],\end{aligned}\quad (\text{S51})$$

we obtain

$$\hat{M}^{(0)} = P(\theta) \frac{\exp\left(-\frac{\lambda_0 s^2}{2T_s(\theta)}\right)}{\sqrt{2\pi T_s(\theta)/\lambda_0}} + [\text{exponentially decaying terms}], \quad (\text{S52})$$

where

$$T_s(\theta) = \frac{T_h \gamma \phi'(\theta)^2 + T_c \Gamma}{\Gamma + \gamma \phi'(\theta)^2}. \quad (\text{S53})$$

It immediately follows that the $O(\epsilon^{-1/2})$ term of $\hat{\Omega}$ vanishes. The sub-leading order of Eq. (S48) becomes

$$\begin{aligned}\frac{\partial \hat{M}^{(1)}}{\partial \mathcal{T}} &= -\frac{\partial}{\partial \theta} \left[\frac{\lambda_0}{\Gamma} s \phi'(\theta) \hat{M}^{(0)} + \frac{T_h}{\Gamma} \phi'(\theta) \frac{\partial}{\partial s} \hat{M}^{(0)} \right] + \phi'(\theta) \frac{\partial}{\partial s} \left[\frac{f}{\Gamma} \hat{M}^{(0)} - \frac{T_h}{\Gamma} \frac{\partial}{\partial \theta} \hat{M}^{(0)} \right] - \frac{\partial}{\partial s} \left[-\frac{k}{\gamma} \phi(\theta) \hat{M}^{(0)} \right] \\ &\quad - \frac{1}{\Gamma \gamma} \frac{\partial}{\partial s} \left[-\lambda_0 s [\Gamma + \gamma \phi'(\theta)^2] \hat{M}^{(1)} - [T_h \gamma \phi'(\theta)^2 + T_c \Gamma] \frac{\partial}{\partial s} \hat{M}^{(1)} \right],\end{aligned}\quad (\text{S54})$$

which gives

$$\begin{aligned}\hat{M}^{(1)} &= -\frac{\Gamma \gamma}{T_h \gamma \phi'(\theta)^2 + T_c \Gamma} \left\{ -\frac{\partial}{\partial \theta} \left[-\frac{T_s(\theta)}{\Gamma} \phi'(\theta) + \frac{T_h}{\Gamma} \phi'(\theta) \right] + \frac{f}{\Gamma} \phi'(\theta) + \frac{k}{\gamma} \phi(\theta) \right. \\ &\quad \left. - \left[-\frac{T_s(\theta)}{\Gamma} \phi'(\theta) + 2 \frac{T_h}{\Gamma} \phi'(\theta) \right] \left[\frac{1}{P(\theta)} \frac{\partial P(\theta)}{\partial \theta} + \frac{T_s'(\theta)}{2T_s(\theta)} \left(\frac{\lambda_0 s^2}{3T_s(\theta)} - 1 \right) \right] \right\} s P(\theta) \frac{\exp\left(-\frac{\lambda_0 s^2}{2T_s(\theta)}\right)}{\sqrt{2\pi T_s(\theta)/\lambda_0}} \\ &\quad + [\text{exponentially decaying terms}].\end{aligned}\quad (\text{S55})$$

Substitution of Eq. (S55) into Eq. (S50) results in

$$\begin{aligned}\hat{\Omega}[P(\theta; t)] &:= -\frac{\partial}{\partial \theta} \left[\frac{f}{\Gamma} P(\theta) - \frac{T_h}{\Gamma} \frac{\partial}{\partial \theta} P(\theta) \right] - \frac{\partial}{\partial \theta} \left[\int ds \frac{\lambda_0}{\Gamma} s \phi'(\theta) \hat{M}^{(1)}[P(\theta'; t)](\theta, s) \right] \\ &= -\frac{\partial}{\partial \theta} \left[-\frac{T_{\text{eff}}(\theta)}{\mathcal{G}(\theta)} \frac{\partial}{\partial \theta} P(\theta) \right] - \frac{\partial}{\partial \theta} \left[\frac{f}{\mathcal{G}(\theta)} P(\theta) + \frac{\partial}{\partial \theta} \left(\frac{\gamma \phi'(\theta)^2}{2\mathcal{G}(\theta)^2} \right) (T_h - T_c) P(\theta) - \frac{k}{\mathcal{G}(\theta)} \phi(\theta) \phi'(\theta) P(\theta) \right],\end{aligned}\quad (\text{S56})$$

which gives the Fokker-Planck equation:

$$\begin{aligned}\frac{\partial P(\theta)}{\partial t} &= -\frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial \theta} \left(\frac{\gamma \phi'(\theta)^2}{2\mathcal{G}(\theta)^2} \right) (T_h - T_c) P(\theta) + \frac{1}{\mathcal{G}(\theta)} (f - k \phi(\theta) \phi'(\theta)) P(\theta) - \frac{T_{\text{eff}}(\theta)}{\mathcal{G}(\theta)} \frac{\partial}{\partial \theta} P(\theta) \right] \\ &= -\frac{\partial}{\partial \theta} \left[-\frac{1}{\mathcal{G}(\theta)} \frac{\partial \ln \mathcal{G}(\theta)}{\partial \theta} (T_h - T_c) P(\theta) + \frac{1}{\mathcal{G}(\theta)} (f - \phi(\theta) \phi'(\theta)) P(\theta) - \frac{1}{\mathcal{G}(\theta)} \frac{\partial}{\partial \theta} (T_{\text{eff}}(\theta) P(\theta)) \right],\end{aligned}\quad (\text{S57})$$

corresponding to Langevin equation

$$\mathcal{G} \odot \dot{\theta} = f - k \phi(\theta) \phi'(\theta) - \frac{\partial \ln \mathcal{G}(\theta)}{\partial \theta} (T_h - T_c) + \sqrt{2\mathcal{G}(\theta)} \odot \sqrt{T_{\text{eff}}(\theta)} \cdot \Xi. \quad (\text{Model-7})$$

We note that Magnasco and Stolovitzky in the investigation of such coarse-graining, derived the dynamics with the same θ -dependent temperature through a phenomenological argument [M. O. Magnasco and G. Stolovitzky, *J. Stat. Phys.* **93**, 615 (1998).], but did not arrive at the inhomogeneous friction and the temperature difference-dependent force (third term of right hand side of Model-7).

S2. ASYMPTOTIC BEHAVIOR OF ENTROPY PRODUCTION RATES

A. Entropy Production Rate in Model-1

Based on the result of §S1, we evaluate the ensemble average of the entropy production rate defined in Model-1, in the limit of $\varepsilon \rightarrow 0$. To consider the ensemble average with respect to $M[P(\theta', p'); \mathcal{T}](\theta, p, s)$, we transform x to $\phi(\theta) + \varepsilon^{1/2}s$:

$$\begin{aligned}
\sigma_1 &= -\frac{1}{T_h} \left(\dot{p} + \frac{\partial U(\theta, x)}{\partial \theta} - f \right) \circ \frac{p}{m} - \frac{1}{T_c} \frac{\partial U(\theta, x)}{\partial x} \circ \dot{x} \\
&= -\frac{1}{T_h} \left(\dot{p} - \varepsilon^{-1/2} \phi'(\theta) \frac{\partial U_I(s)}{\partial s} - f \right) \circ \frac{p}{m} - \frac{1}{T_c} \left(\varepsilon^{-1/2} \frac{\partial U_I(s)}{\partial s} + \frac{\partial U_0(\phi(\theta))}{\partial \phi(\theta)} + O(\varepsilon^{1/2}) \right) \circ (\phi'(\theta) \dot{\theta} + \varepsilon^{1/2} \dot{s}) \\
&= -\frac{1}{T_h} \left(\dot{p} - \varepsilon^{-1/2} \phi'(\theta) \frac{\partial U_I(s)}{\partial s} - f \right) \circ \frac{p}{m} - \frac{1}{T_c} \left(\varepsilon^{-1/2} \frac{\partial U_I(s)}{\partial s} \circ (\phi'(\theta) \dot{\theta} + \varepsilon^{1/2} \dot{s}) + \frac{\partial U_0(\phi(\theta))}{\partial \phi(\theta)} \phi'(\theta) \dot{\theta} + O(\varepsilon^{1/2}) \right) \\
&= \left[-\frac{1}{T_h} (\dot{p} - f) \circ \frac{p}{m} - \frac{1}{T_c} \left(\frac{\partial U_I(s)}{\partial s} \circ \dot{s} + \frac{\partial U_0(\phi(\theta))}{\partial \phi(\theta)} \phi'(\theta) \dot{\theta} \right) \right] + \varepsilon^{-1/2} \left[\frac{1}{T_h} \phi'(\theta) \frac{\partial U_I(s)}{\partial s} - \frac{1}{T_c} \left(\frac{\partial U_I(s)}{\partial s} \circ \phi'(\theta) \dot{\theta} \right) \right] \\
&\quad + O(\varepsilon^{1/2}), \tag{S58}
\end{aligned}$$

where we collected the terms up to $O(1)$, since the ensemble average of the $O(\varepsilon^{-1/2})$ terms with respect to $M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s)$ decays exponentially with time. The ensemble average of the $O(1)$ terms with respect to $M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s)$ gives

$$\langle [\text{first term of Eq. (S58)}] \rangle = \left\langle -\frac{1}{T_h} (\dot{p} - f) \circ \frac{p}{m} - \frac{1}{T_c} \frac{\partial U_0(\phi(\theta))}{\partial \phi(\theta)} \phi'(\theta) \dot{\theta} \right\rangle \tag{S59}$$

where $\langle -U'_I(s) \circ \dot{s}/T_c \rangle$ vanishes by assuming that $U_I(s)$ and T_c have no explicit dependence on time¹. The ensemble average of the $O(\varepsilon^{-1/2})$ terms with respect to $M^{(0)}[P(\theta', p'); \mathcal{T}](\theta, p, s)$ is essentially zero, whereas the average with respect to $M^{(1)}[P(\theta', p'); \mathcal{T}](\theta, p, s)$ gives

$$\begin{aligned}
\langle [\text{second term of Eq. (S58)}] \rangle &= \left\langle \left[\frac{1}{T_h} \frac{p}{m} - \frac{1}{T_c} \dot{\theta} \right] \varepsilon^{-1/2} \phi'(\theta) \frac{\partial U_I(s)}{\partial s} \right\rangle \\
&= \int d\theta dp ds \left(\frac{1}{T_h} - \frac{1}{T_c} \right) \phi'(\theta) \frac{p}{m} \frac{\partial U_I(s)}{\partial s} M^{(1)}[P(\theta', p'); \mathcal{T}](\theta, p, s) \\
&= \int d\theta dp \left(\frac{1}{T_h} - \frac{1}{T_c} \right) \phi'(\theta) \frac{p}{m} \gamma \left\{ -T_c \phi'(\theta) \frac{\partial P(\theta, p)}{\partial p} - \left[\phi'(\theta) \frac{p}{m} + \frac{U'_0(\phi(\theta))}{\gamma} \right] P(\theta, p) \right\} \\
&= \left\langle \left(\frac{1}{T_h} - \frac{1}{T_c} \right) \phi'(\theta) \frac{\gamma}{m} \left\{ T_c \phi'(\theta) - \left[\phi'(\theta) \frac{p^2}{m} + \frac{U'_0(\phi(\theta))p}{\gamma} \right] \right\} \right\rangle. \tag{S60}
\end{aligned}$$

¹ In fact, Eq. (7) in the main text holds for the difference in the irreversible entropy production even when $U_I(s)$ varies over time.

Therefore, the leading order of the ensemble average of the entropy production rate results in Eq. (7) in the main text as,

$$\begin{aligned}
\langle \sigma_1 \rangle &= \left\langle -\frac{1}{T_h} (\dot{p} - f) \circ \frac{p}{m} - \frac{1}{T_c} \frac{\partial U_0(\phi(\theta))}{\partial \theta} \dot{\theta} \right\rangle + \left\langle \left(\frac{1}{T_h} - \frac{1}{T_c} \right) \frac{\gamma}{m} \left[T_c \phi'(\theta)^2 - \left(\phi'(\theta)^2 \frac{p^2}{m} + \frac{p}{\gamma} \frac{\partial U_0(\phi(\theta))}{\partial \theta} \right) \right] \right\rangle \\
&= \left\langle -\left(\frac{1}{T_{\text{eff}}(\theta)} - \frac{T_h - T_c}{T_h} \frac{\gamma \phi'(\theta)^2}{\mathcal{G}(\theta) T_{\text{eff}}(\theta)} \right) (\dot{p} - f) \circ \frac{p}{m} - \left(\frac{1}{T_{\text{eff}}(\theta)} + \frac{T_h - T_c}{T_c} \frac{\Gamma}{\mathcal{G}(\theta) T_{\text{eff}}(\theta)} \right) \frac{\partial U_0(\phi(\theta))}{\partial \theta} \frac{p}{m} \right\rangle \\
&\quad + \left\langle \left(\frac{1}{T_h} - \frac{1}{T_c} \right) \left[\frac{\gamma \phi'(\theta)^2}{m} \left(T_c - \frac{p^2}{m} \right) - \frac{\partial U_0(\phi(\theta))}{\partial \theta} \frac{p}{m} \right] \right\rangle \\
&= \left\langle -\frac{1}{T_{\text{eff}}(\theta)} \left(\dot{p} + \frac{\partial U_0(\phi(\theta))}{\partial \theta} - f \right) \circ \frac{p}{m} \right\rangle \\
&\quad + \left\langle \frac{T_h - T_c}{T_h} \frac{\gamma \phi'(\theta)^2}{\mathcal{G}(\theta) T_{\text{eff}}(\theta)} (\dot{p} - f) \circ \frac{p}{m} - \left(\frac{T_h - T_c}{T_c} \frac{\Gamma}{\mathcal{G}(\theta) T_{\text{eff}}(\theta)} + \frac{1}{T_h} - \frac{1}{T_c} \right) \frac{\partial U_0(\phi(\theta))}{\partial \theta} \frac{p}{m} \right\rangle \\
&\quad + \left\langle \left(\frac{1}{T_h} - \frac{1}{T_c} \right) \frac{\gamma \phi'(\theta)^2}{m} \left(T_c - \frac{p^2}{m} \right) \right\rangle \\
&= \left\langle -\frac{1}{T_{\text{eff}}(\theta)} \left(\dot{p} + \frac{\partial U_0(\phi(\theta))}{\partial \theta} - f \right) \circ \frac{p}{m} \right\rangle + \left\langle \frac{T_h - T_c}{T_h} \frac{\gamma \phi'(\theta)^2}{\mathcal{G}(\theta) T_{\text{eff}}(\theta)} \left(\dot{p} + \frac{\partial U_0(\phi(\theta))}{\partial \theta} - f \right) \circ \frac{p}{m} \right\rangle \\
&\quad + \left\langle \left(\frac{1}{T_h} - \frac{1}{T_c} \right) \frac{\gamma \phi'(\theta)^2}{m} \left(T_c - \frac{p^2}{m} \right) \right\rangle \\
&= \left\langle -\frac{1}{T_{\text{eff}}(\theta)} \left(\dot{p} + \frac{\partial U_0(\phi(\theta))}{\partial \theta} - f \right) \circ \frac{p}{m} \right\rangle + \left\langle \frac{\Gamma \gamma \phi'(\theta)^2}{\mathcal{G}(\theta) T_{\text{eff}}(\theta)} \left(\frac{1}{T_h} - \frac{1}{T_c} \right) (T_c - T_h) \frac{p^2}{m^2} \right\rangle, \tag{S61}
\end{aligned}$$

where, in the last line, we used

$$\left\langle -\frac{\gamma \phi'(\theta)^2}{\mathcal{G}(\theta) T_{\text{eff}}(\theta)} \left(\dot{p} + \frac{\partial U_0(\phi(\theta))}{\partial \theta} - f \right) \circ \frac{p}{m} \right\rangle = \left\langle \frac{\gamma \phi'(\theta)^2}{m T_{\text{eff}}(\theta)} \left(\frac{p^2}{m} - T_{\text{eff}}(\theta) \right) \right\rangle \tag{S62}$$

which follows from the underdamped Langevin equation of Model-2.

B. Entropy Production Rate in Model-2

Next, we evaluate the ensemble average of the entropy production rate defined in Model-2, in the limit of $m/\Gamma \rightarrow 0$. The entropy production rate may be rewritten as

$$\begin{aligned}
\sigma_2 &= \frac{1}{T_{\text{eff}}(\theta)} \left(\dot{p} + \frac{\partial U_0(\phi(\theta))}{\partial \theta} - f \right) \circ \frac{p}{m} = -\frac{1}{T_{\text{eff}}(\theta)} \left[\frac{d}{dt} \frac{p^2}{2m} + \left(\frac{\partial U_0(\phi(\theta))}{\partial \theta} - f \right) \frac{p}{m} \right] \\
&= -\frac{d}{dt} \left(\frac{1}{T_{\text{eff}}(\theta)} \frac{p^2}{2m} \right) - \frac{p^2}{2m} \frac{1}{T_{\text{eff}}(\theta)^2} \frac{p}{m} \frac{\partial}{\partial \theta} T_{\text{eff}}(\theta) - \frac{1}{T_{\text{eff}}(\theta)} \left(\frac{\partial U_0(\phi(\theta))}{\partial \theta} - f \right) \frac{p}{m}, \tag{S63}
\end{aligned}$$

where we assume that $T_{\text{eff}}(\theta)$ does not depend on time explicitly². Since the ensemble average with respect to $\tilde{M}^{(0)}[P(\theta'); \tau](\theta, \tilde{p})$ vanishes, we obtain a finite contribution from that with respect to $\tilde{M}^{(1)}[P(\theta'); \tau](\theta, \tilde{p})$ as

$$\begin{aligned}
\langle \sigma_2 \rangle &= \int d\theta \left\{ \frac{1}{\mathcal{G}(\theta) T_{\text{eff}}(\theta)} \left[-T_{\text{eff}}(\theta) \frac{\partial P(\theta)}{\partial \theta} + \left(-\frac{\partial U_0(\phi(\theta))}{\partial \theta} - T'_{\text{eff}}(\theta) + f \right) P(\theta) \right] \right. \\
&\quad \cdot \left. \left(-\frac{\partial U_0(\phi(\theta))}{\partial \theta} - \frac{3}{2} T'_{\text{eff}}(\theta) + f \right) + \frac{T_{\text{eff}}(\theta)}{2\mathcal{G}(\theta)} \left(\frac{T'_{\text{eff}}(\theta)}{T_{\text{eff}}(\theta)} \right)^2 P(\theta) \right\} \\
&= \left\langle \frac{1}{T_{\text{eff}}(\theta)} \left(-\frac{\partial U_0(\phi(\theta))}{\partial \theta} - \frac{3}{2} T'_{\text{eff}}(\theta) + f \right) \circ \dot{\theta} + \frac{T_{\text{eff}}(\theta)}{2\mathcal{G}(\theta)} \left(\frac{T'_{\text{eff}}(\theta)}{T_{\text{eff}}(\theta)} \right)^2 \right\rangle. \tag{S64}
\end{aligned}$$

In order to obtain the last expression, we used the overdamped Langevin equation of Model-3.

² In addition to Eq. (7), we can derive Eq. (8) for the difference in the irreversible entropy production even when $T_{\text{eff}}(\theta)$ varies over time.

C. Entropy Production Rate in Model-6

We here show that the ensemble average of the entropy production rate diverges in Model-6 when we take the tightly confined limit. Since the entropy production rate for Model-6 is written as

$$\sigma_6 := \frac{Q_6^h}{T_h} + \frac{Q_6^c}{T_c} := -\frac{1}{T_h} \left[\frac{\partial U(\theta, x)}{\partial \theta} - f \right] \circ \dot{\theta} - \frac{1}{T_c} \frac{\partial U(\theta, x)}{\partial x} \circ \dot{x}, \quad (\text{S65})$$

the ensemble averages of heat fluxes, Q_6^h and Q_6^c , with respect to $\hat{M}^{(0)}$ gives

$$\begin{aligned} \langle Q_6^h \rangle &= \left\langle - \left[\frac{\partial U(\theta, x)}{\partial \theta} - f \right] \circ \dot{\theta} \right\rangle = \langle [\lambda(x - \phi(\theta))\phi'(\theta) + f] \circ \dot{\theta} \rangle \\ &= \frac{\lambda}{\Gamma} \langle \lambda(x - \phi(\theta))^2 \phi'(\theta)^2 - T_h [\phi'(\theta)^2 - (x - \phi(\theta))\phi''(\theta)] \rangle + \langle f\dot{\theta} \rangle \\ &= \frac{\lambda_0}{\epsilon\Gamma} \langle \lambda_0 s^2 \phi'(\theta)^2 + T_h [\epsilon^{1/2} s \phi''(\theta) - \phi'(\theta)^2] \rangle + \langle f\dot{\theta} \rangle = \frac{\lambda_0}{\epsilon\Gamma} \langle T_s(\theta)\phi'(\theta)^2 - T_h \phi'(\theta)^2 \rangle + \langle f\dot{\theta} \rangle \\ &= -\frac{\lambda_0 \phi'(\theta)^2}{\epsilon\mathcal{G}(\theta)} (T_h - T_c) + \langle f\dot{\theta} \rangle, \end{aligned} \quad (\text{S66})$$

$$\begin{aligned} \langle Q_6^c \rangle &= \left\langle - \frac{\partial U(\theta, x)}{\partial x} \circ \dot{x} \right\rangle = \langle -[kx + \lambda(x - \phi(\theta))] \circ \dot{x} \rangle = \frac{1}{\gamma} \langle [kx + \lambda(x - \phi(\theta))]^2 - (k + \lambda)T_c \rangle \\ &= \frac{1}{\gamma} \langle [k(\epsilon^{1/2} + \phi(\theta)) + \lambda_0 \epsilon^{-1/2} s]^2 - (k + \lambda_0 \epsilon^{-1})T_c \rangle = \frac{\lambda_0}{\gamma\epsilon} \langle \lambda_0 s^2 - T_c + O(\epsilon) \rangle = \frac{\lambda_0}{\gamma\epsilon} \langle T_s(\theta) - T_c + O(\epsilon) \rangle \\ &= \frac{\lambda_0 \phi'(\theta)^2}{\epsilon\mathcal{G}(\theta)} (T_h - T_c) + O(\epsilon^0). \end{aligned} \quad (\text{S67})$$

Therefore, in the limit of $\lambda/\gamma \rightarrow \infty$ ($\epsilon \rightarrow 0$), the entropy production rate diverges, unless $T_h = T_c$.

S3. ϵ -DEPENDENCE OF MAXIMAL EFFICIENCIES

In FIG. S2, we show the ϵ -dependence of maximal efficiency obtained from the fitting of torque-efficiency curves. As seen from the figure, η_1 and η_2 converge to zero and a finite value, respectively. These results support our observation that, in the limit of $\epsilon \rightarrow 0$, η_1 vanishes irrespective of f , and η_2 converges to a certain torque-dependent curve.

S4. DETAILS OF NUMERICAL SIMULATION

We here describe the details of numerical simulations. The numerical simulations are mainly carried out based on the Langevin equation of Model-1. In the numerical integration of Langevin equation, we employ the velocity Verlet method for the underdamped part and the Euler method for the overdamped part. The time step is set to 2×10^{-3} and the total length of simulations is set to 2^{12} . The ensemble averages of the entropy production are calculated from 2^{12} -independent runs, and the average entropy production rates are obtained from linear fitting.

In the numerical investigation of efficiency (Fig. 4, S1), we use the numerical integration of the Kramers equation of Model-2 together with the Langevin equation of Model-1. The phase space with a cut-off of momentum at $p = \pm 8$ is discretized into $2^8 \times (2^7 + 1)$ elements along the position and momentum axes, respectively. The derivatives with respect to θ or p are approximated by the central difference. The time step is set to 0.056×10^{-5} and the total length of simulations is set to 2^3 . In order to calculate η_2 and η_3 , we first evaluate the heat fluxes as functions of θ , $\langle Q_{2,3}(\theta) \rangle$. We then obtain the average rates of heat release to the heat bath at a given effective temperature, $\langle Q_{2,3}(T) \rangle := \int d\theta \langle Q_{2,3}(\theta) \rangle \delta(T_{\text{eff}}(\theta) - T)$. The averaged heat release and absorption rates are defined as

$$\langle Q_{2,3}^{\text{rel}} \rangle = \int dT \langle Q_{2,3}(T) \rangle \Theta(\langle Q_{2,3}(T) \rangle), \quad \langle Q_{2,3}^{\text{abs}} \rangle = - \int dT \langle Q_{2,3}(T) \rangle \Theta(-\langle Q_{2,3}(T) \rangle), \quad (\text{S68})$$

where Θ represents Heaviside step function. η_2 and η_3 are calculated by

$$\eta_{2,3} = 1 - \frac{\langle Q_{2,3}^{\text{ref}} \rangle}{\langle Q_{2,3}^{\text{abs}} \rangle} \quad (\text{S69})$$

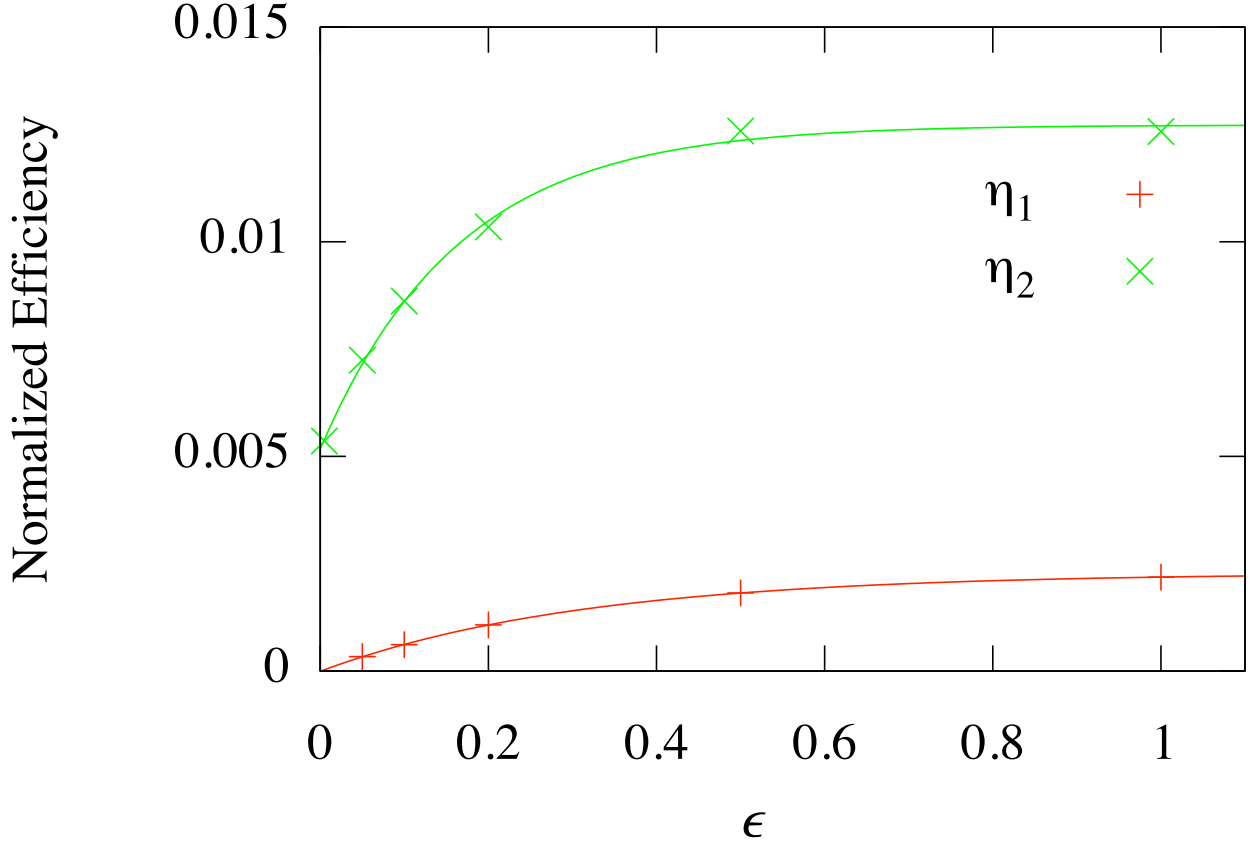


FIG. S2. Maximal efficiency. Maximal value of η_1 and η_2 obtained from the fitting of FIG. 4 by parabolic functions are plotted against ϵ . Solid lines $A \exp(-\epsilon/\epsilon_0) + C$ are also plotted as guides for eye, where A, ϵ_0 and C are fitting parameters. The efficiencies are normalized by the Carnot efficiency of Model-1, $1 - T_c/T_h$.

S5. STOCHASTIC DYNAMICS AND THERMODYNAMICS OF MODEL-4

We first show that Model-4 reproduces Model-2 in the sense of dynamics at the sufficiently long time scale. To describe the stochastic switching of the heat bath that couples to the system, we introduce a new stochastic variable, $i = h, c$. The transition probability of Model-4 is given by

$$W(\theta_{t+\Delta t}, p_{t+\Delta t}, i_{t+\Delta t} | \theta_t, p_t, i_t) = \begin{cases} (1 - \Lambda \Delta t) W(\theta_{t+\Delta t} | \theta_t, p_t) W(p_{t+\Delta t} | \theta_t, p_t, i_t) & (\text{if } i_{t+\Delta t} = i_t) \\ \Lambda \Delta t \delta(\theta_{t+\Delta t} - \theta_t) \delta(p_{t+\Delta t} - p_t) & (\text{if } i_{t+\Delta t} \neq i_t) \end{cases} \quad (\text{S70})$$

$$W(\theta_{t+\Delta t} | \theta_t, p_t) := \delta\left(\theta_{t+\Delta t} - \theta_t - \frac{p_t}{m} \Delta t\right) \quad (\text{S71})$$

$$W(p_{t+\Delta t} | \theta_t, p_t, i_t) := \frac{1}{\sqrt{8\pi\Gamma_{i_t}(\theta_t)T_{i_t}\Delta t}} \exp\left(-\frac{\left[(p_{t+\Delta t} - p_t) + 2\frac{\Gamma_{i_t}(\theta_t)}{m}p_t\Delta t + (U'_{\text{eff}}(\theta_t) - f)\Delta t\right]^2}{8\Gamma_{i_t}(\theta_t)T_{i_t}\Delta t}\right). \quad (\text{S72})$$

where Δt represents infinitesimal interval, and Λ is the switching rate of i . The transition probability from (θ_t, p_t) to $(\theta_{t+N\Delta t}, p_{t+N\Delta t})$ for time $N\Delta t$ is given by tracing out $(\theta_s, p_s)_{s=t+(N-1)\Delta t}^{s=t+N\Delta t}, (i_s)_{s=t}^{s=t+N\Delta t}$ so that

$$\begin{aligned} & W(\theta_{t+N\Delta t}, p_{t+N\Delta t} | \theta_t, p_t) \\ &= \int d\theta_{t+\Delta t} dp_{t+\Delta t} \cdots d\theta_{t+(N-1)\Delta t} dp_{t+(N-1)\Delta t} \sum_{i_t, \dots, i_{t+N\Delta t}} \prod_{n=0}^{N-1} W(\theta_{t+(n+1)\Delta t}, p_{t+(n+1)\Delta t}, i_{t+(n+1)\Delta t} | \theta_{t+n\Delta t}, p_{t+n\Delta t}, i_{t+n\Delta t}) P(i_t | \theta_t, p_t). \end{aligned} \quad (\text{S73})$$

By choosing N so that $N\Delta t$ is shorter than the time scales where θ and p change (since Δt is infinitesimal, it is always possible), we may approximate Eq. (S72) and (S71) by

$$W(p_{t+(n+1)\Delta t}|\theta_{t+n\Delta t}, p_{t+n\Delta t}, i_{t+n\Delta t}) \approx \frac{1}{2} \frac{1}{\sqrt{8\pi\Gamma_{i_{t+n\Delta t}}(\theta_t)T_{i_{t+n\Delta t}}\Delta t}} \exp\left(-\frac{\left[(p_{t+(n+1)\Delta t} - p_{t+n\Delta t}) + 2\frac{\Gamma_{i_{t+n\Delta t}}(\theta_t)}{m}p_t\Delta t + (U'_{\text{eff}}(\theta_t) - f)\Delta t\right]^2}{8\Gamma_{i_{t+n\Delta t}}(\theta_t)T_{i_{t+n\Delta t}}\Delta t}\right) \quad (\text{S74})$$

$$W(\theta_{t+(n+1)\Delta t}|\theta_{t+n\Delta t}, p_{t+n\Delta t}) = \delta\left(\theta_{t+(n+1)\Delta t} - \theta_{t+n\Delta t} - \frac{p_{t+n\Delta t}}{m}\Delta t\right) \approx \delta\left(\theta_{t+(n+1)\Delta t} - \theta_{t+n\Delta t} - \frac{p_t}{m}\Delta t\right). \quad (\text{S75})$$

Under this approximation, Eq. (S73) may be rewritten as

$$W(\theta_{t+N\Delta t}, p_{t+N\Delta t}|\theta_t, p_t) = \delta\left(\theta_{t+N\Delta t} - \theta_t - \frac{p_t}{m}N\Delta t\right) W(p_{t+N\Delta t}|\theta_t, p_t), \quad (\text{S76})$$

and, since $p_{t+(n+1)\Delta t} - p_{t+n\Delta t}$ is identically distributed and its correlation through i decays with $1/\Lambda$, $p_{t+N\Delta t} - p_t$ follows a Gaussian distribution when $N\Delta t \gg 1/\Lambda$. The average and variance calculated with respect to the transition probability $W(p_{t+N\Delta t}|\theta_t, p_t)$ is

$$\begin{aligned} \overline{(p_{t+N\Delta t} - p_t)} &= \sum_n \overline{(p_{t+(n+1)\Delta t} - p_{t+n\Delta t})} \\ &= -\frac{\Gamma_h(\theta_t) + \Gamma_c(\theta_t)}{m} p_t N\Delta t - (U'_{\text{eff}}(\theta_t) - f)N\Delta t = -\frac{\mathcal{G}(\theta_t)}{m} p_t N\Delta t - (U'_{\text{eff}}(\theta_t) - f)N\Delta t, \end{aligned} \quad (\text{S77})$$

$$\overline{(p_{t+N\Delta t} - p_t)^2} = 2[\Gamma_h(\theta_t)T_h + \Gamma_c(\theta_t)T_c]N\Delta t + O[N(\Delta t)^2, N\Delta t/\Lambda] = 2\mathcal{G}(\theta_t)T_{\text{eff}}(\theta_t)\Delta t + O[(\Delta t)^2, N\Delta t/\Lambda]. \quad (\text{S78})$$

Therefore, we obtain the transition probability equivalent to that of Model-2:

$$W(\theta_{t+N\Delta t}, p_{t+N\Delta t}|\theta_t, p_t) = \delta\left(\theta_{t+N\Delta t} - \theta_t - \frac{p_t}{m}N\Delta t\right) \frac{1}{\sqrt{4\pi\mathcal{G}(\theta_t)T_{\text{eff}}(\theta_t)N\Delta t}} \exp\left(-\frac{\left[(p_{t+N\Delta t} - p_t) + \frac{\mathcal{G}(\theta_t)}{m}p_t N\Delta t + (U'_{\text{eff}}(\theta_t) - f)N\Delta t\right]^2}{4\mathcal{G}(\theta_t)T_{\text{eff}}(\theta_t)N\Delta t}\right). \quad (\text{S79})$$

Next, we show Eq. (10) in the main text. In Model-4, the ensemble average of the entropy production rate defined through the transition probability is

$$\langle\sigma_4\rangle = \int d\theta_t dp_t d\theta_{t+\Delta t} dp_{t+\Delta t} \sum_{i_t=h,c, i_{t+\Delta t}=h,c} W(\theta_{t+\Delta t}, p_{t+\Delta t}, i_{t+\Delta t}|\theta_t, p_t, i_t) P(\theta_t, p_t, i_t) \ln \frac{W(\theta_{t+\Delta t}, p_{t+\Delta t}, i_{t+\Delta t}|\theta_t, p_t, i_t)}{W(\theta_t, -p_t, i_t|\theta_{t+\Delta t}, -p_{t+\Delta t}, i_{t+\Delta t})}, \quad (\text{S80})$$

where $P(\theta_t, p_t)$ is the steady state probability density. Since, in the case of $i_{t+\Delta t} = i_t$, the logarithmic term corresponds to the entropy productions of constituent Langevin equations of Model-4, we may rewrite, in the limit of $\Delta t \rightarrow 0$

$$\begin{aligned} \langle\sigma_4\rangle &= \int d\theta_t dp_t d\theta_{t+\Delta t} dp_{t+\Delta t} \sum_{i=h,c} W(\theta_{t+\Delta t}, p_{t+\Delta t}|\theta_t, p_t; i) P(\theta_t, p_t) \left[-\frac{1}{T_i} \left(\frac{p_{t+\Delta t} - p_t}{\Delta t} + U'_{\text{eff}}(\theta) - f \right) \frac{p_t + p_{t+\Delta t}}{2m} \right] \\ &= \int d\theta_t dp_t P(\theta_t, p_t) \sum_{i=h,c} \frac{\Gamma_i(\theta_t)}{m} \left(\frac{p_t^2}{mT_i} - 1 \right) = \left\langle \frac{\Gamma_h}{m} \left(\frac{p^2}{mT_h} - 1 \right) + \frac{\Gamma_c(\theta)}{m} \left(\frac{p^2}{mT_c} - 1 \right) \right\rangle. \end{aligned} \quad (\text{S81})$$

By comparing with the expression $\lim_{\lambda/\gamma \rightarrow \infty} \langle\sigma_2\rangle = \left\langle \frac{\mathcal{G}(\theta)}{m} \left(\frac{p^2}{mT_{\text{eff}}(\theta)} - 1 \right) \right\rangle$, we arrive at

$$\langle\sigma_4\rangle = \lim_{\lambda/\gamma \rightarrow \infty} \langle\sigma_2\rangle + \left\langle \frac{\Gamma(\mathcal{G}(\theta) - \Gamma)}{\mathcal{G}(\theta)T_{\text{eff}}(\theta)} \left(\frac{1}{T_h} - \frac{1}{T_c} \right) (T_h - T_c) \frac{p^2}{m^2} \right\rangle = \lim_{\lambda/\gamma \rightarrow \infty} \langle\sigma_1\rangle. \quad (\text{S82})$$